ex Show that $L: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$L(x) = x_2 e_2$ is a Linear operator

$$L(x) = L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

sol $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$

* $L(x+y) = L\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 + y_2 \end{bmatrix}$

$L(x) = L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$

$L(y) = L\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ y_2 \end{bmatrix}$

$\Rightarrow L(x) + L(y) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 + y_2 \end{bmatrix}$

$\therefore L(x+y) = L(x) + L(y)$

** $L(\alpha x) = L\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = L\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha x_2 \end{bmatrix}$

$\alpha L(x) = \alpha L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha x_2 \end{bmatrix}$

$\therefore L(\alpha x) = \alpha L(x)$

Thus L is a Linear operator on $\mathbb{R}^2$

Sec. 4.1 + 4.2

ex $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by

$$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$ show that L is not L.T

sol we can use the result that If L is L.T then

$$L(\underset{v}{0}) = 0_w$$

bo $L\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq 0_w$ so it is not L.T

or $L(x+y) = L\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ 1 \end{bmatrix}$

but $L(x)+L(y) = L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + L\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$= \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ 2 \end{bmatrix}$$

not equal

or $L(\alpha x) = L\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ 1 \end{bmatrix}$

$\alpha L(x) = \alpha\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha \end{bmatrix}$

not equal.

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Let $L: \mathbb{R}^2 \to \mathbb{R}^2$

and $L\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix}$

$L\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix}$

find the value of $L\begin{bmatrix}7\\5\end{bmatrix}$

Sol Write $\begin{bmatrix}7\\5\end{bmatrix}$ as $\alpha\begin{bmatrix}1\\2\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix}$

Let $\alpha\begin{bmatrix}1\\2\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}7\\5\end{bmatrix}$

Then: $\alpha + \beta = 7$

$2\alpha - \beta = 5$

$3\alpha = 12$

$\boxed{\alpha = 4}$

so $\beta = 7 - \alpha = 7 - 4 = 3$

$\boxed{\beta = 3}$

Thus: $\begin{bmatrix}7\\5\end{bmatrix} = 4\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}1\\-1\end{bmatrix}$

So $L\begin{bmatrix}7\\5\end{bmatrix} = L\left[4\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}1\\-1\end{bmatrix}\right)$

$= 4L\begin{bmatrix}1\\2\end{bmatrix} + 3L\begin{bmatrix}1\\-1\end{bmatrix}$

$= 4\begin{bmatrix}-2\\3\end{bmatrix} + 3\begin{bmatrix}5\\2\end{bmatrix}$

$= \begin{bmatrix}7\\18\end{bmatrix}$

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The Image and kernel of L:

Def ① $L: V \longrightarrow W$ is L.T Then

kernel of L denoted Ker(L) is

$$Ker(L) = \{ v \in V ; L(v) = 0_w \}$$

and Ker(L) is a subspace of V

Def ② $L: V \longrightarrow W$ is L.T Then

the image of a set S of V is

$$L(S) = \{ w \in W \mid w = L(v) \text{ for some } v \in V \}$$

(*) If $S = V \Rightarrow L(V) = \text{range}(L)$

(*) If S is a subspace of V then L(S) is a subspace of W

Thus Rang(L) is a subspace of W

ex 1 $\quad L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is L.O

$$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

or $\quad L(X) = x_1 e_1$

find (1) Ker (L)
(2) Rang (L)

sol $\quad$ Ker(L) :

$$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{matrix} x_1 = 0 \\ x_2 = \alpha \end{matrix}$$

$$Ker(L) = \left\{ \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\} = span(e_2)$$

$$Range(L) = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, x_1 \in \mathbb{R} \right\}$$

$$= \left\{ x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_1 \in \mathbb{R} \right\}$$

$$= span(e_1).$$

Ex 2 $L: \mathbb{R}^3 \to \mathbb{R}^2$ is L.T given by

$$L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$$

Let $S = \text{span}(e_1, e_3)$

find (1) $\ker L$, (3) $L(S)$, (2) $\text{Image}(L)$

Sol $\boxed{\ker(L)}$:- $L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \bar{0}$

$$\begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\ker(L) = \{ v \in V : L(v) = 0 \}$

We have the system: $\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$

$$\equiv \begin{array}{cc} x_1 & x_2 \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$x_3 = \alpha \quad , \quad \alpha \in \mathbb{R}$

$x_2 + x_3 = 0 \Rightarrow x_2 = -\alpha$

$x_1 + x_2 = 0 \Rightarrow x_1 = \alpha$

Thus $\ker(L) = \left\{ \begin{bmatrix} \alpha \\ -\alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\}$

$= \left\{ \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$

$= \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$

$\ker(L)$ is a subspace of $\mathbb{R}^3$

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$\boxed{2}$ $\text{Image}(L) = \text{Range}(L)$

$$= \left\{ \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} ,\ x_1, x_2, x_3 \in \mathbb{R} \right\}$$

$$= \left\{ x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},\ x_1, x_2, x_3 \in \mathbb{R}$$

$$= \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \text{span}(e_1, e_2)$$

$$= \mathbb{R}^2$$

$\boxed{3}$ $S = \text{span}(e_1, e_3) = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,\ \alpha, \beta \in \mathbb{R} \right\}$

$$= \left\{ \begin{bmatrix} \alpha \\ 0 \\ \beta \end{bmatrix} ,\ \alpha, \beta \in \mathbb{R} \right\}$$

$$L(S) = \left\{ \begin{bmatrix} \alpha + 0 \\ 0 + \beta \end{bmatrix} ,\ \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} ,\ \alpha, \beta \in \mathbb{R} \right\}$$

$$= \mathbb{R}^2$$

ex: Find The Kernel and the range of the operator on $\mathbb{R}^3$ defined by:

$$L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix}$$

sol: Ker(L):

$$\begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = \alpha \\ x_3 = \beta \end{array}$$

$$\ker(L) = \left\{ \begin{bmatrix} 0 \\ \alpha \\ \beta \end{bmatrix}, \ \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \alpha, \beta \in \mathbb{R} \right\}$$

$$= \text{span}(e_2, e_3)$$

Range of L

$$= \left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix}, \ x_1 \in \mathbb{R} \right\}$$

$$= \left\{ x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ x_1 \in \mathbb{R} \right\}$$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Sec 4.2 Matrix Representation of L.T

Let A be $m\times n$ matrix and $X \in \mathbb{R}^n$ then

$L_A(X) = AX$ is a L.T

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\underset{n\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} \longrightarrow \underset{m\times n}{\begin{bmatrix} & A & \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}$$

$$= \underset{m\times 1}{\begin{bmatrix} AX \end{bmatrix}} \in \mathbb{R}^m$$

Result For each L.T

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

We can find a matrix A of order $m\times n$

such that

$L(x) = L_A(x) = AX$

Theorem: If $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a L.T, then there is a matrix A of order $m\times n$ with $L(X) = L_A(X) = AX$ for each $X \in \mathbb{R}^n$ which is given by:

$$A = \begin{bmatrix} L(e_1) & L(e_2) & \cdots & L(e_n) \end{bmatrix}$$

Where $e_1, \ldots, e_n \in \mathbb{R}^n$

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ex Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$L(x) = (x_1 + x_2 \,,\, x_2 + x_3)^T$$

find) the matrix representation of L

sol $L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$

$\mathbb{R}^3$: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Thus : $A = \begin{bmatrix} L(e_1) & L(e_2) & L(e_3) \end{bmatrix}$

$L(e_1) = \cancel{L}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$L(e_2) = L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$L(e_3) = L\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\underset{2\times 3}{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $L(x) = AX$

So : $L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$

Notes

$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is L.T

and $\underset{n\times 1}{L(x)} = \underset{m\times n\; n\times 1}{AX}$ then

$\boxed{1}$ $\operatorname{Ker}(L) = N(A)$

$\boxed{2}$ $\operatorname{Range}(L) =$ column space of (A)

for the given example:

$\operatorname{Ker}(L) = N(A)$

$$A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \Longrightarrow$$ The augmented matrix of the system $AX = 0$ is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

x_3 is free $\Rightarrow x_3 = \alpha$

$x_2 + x_3 = 0 \Rightarrow x_2 = -\alpha$

$x_1 + x_2 = 0 \Rightarrow x_1 = \alpha$

$$N(A) = \left\{ \begin{bmatrix} \alpha \\ -\alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) = \operatorname{Ker}(L)$$

$$\begin{aligned} \operatorname{Range}(L) &= \text{Column space of } (A) \\ &= \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

or $\operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$