Section 3.3 Linear Independence

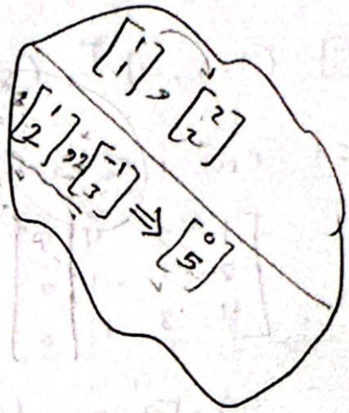
Def Let V be a vector space,
Let $v_1, v_2, \dots, v_n$ be vectors in V, then $v_1, \dots, v_n$ are said to be

1. Linearly Independent if   L. Ind

$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \bar{0}$

then $\boxed{c_1 = c_2 = \dots = c_n = 0}$

(unique solution)



2. Linearly dependent if   L. D

$c_1 v_1 + \dots + c_n v_n = \bar{0}$

for $c_1, c_2, \dots, c_n$ not all are zero
(Infinite number of solution).

ex Are $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$ L. D or L. Ind.

sol $c_1 v_1 + c_2 v_2 = \bar{0}$

$c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\left[\begin{array}{cc|c}1&1&0\\1&2&0\end{array}\right] \Rightarrow \begin{array}{c}\begin{array}{cc}c_1 & c_2\end{array}\\ \left[\begin{array}{cc|c}1&1&0\\0&1&0\end{array}\right]\end{array}$ has the solution $c_1 = c_2 = 0$ unique solution
no free variables.

ex Which of the following collections of vectors are Linearly Independent in $\mathbb{R}^3$

ⓐ $(1,1,1)^T, (1,1,0)^T, (1,0,0)^T$

sol $c_1 v_1 + c_2 v_2 + c_3 v_3 = \bar{0}$

$$\begin{array}{c}\begin{array}{ccc}c_1 & c_2 & c_3\end{array}\\ \left[\begin{array}{ccc|c}1&1&1&0\\1&1&0&0\\1&0&0&0\end{array}\right]\end{array}$$

$c_3 = 0$
$c_2 = 0$
$c_1 = 0$
$\Downarrow$
unique sol.
$\Rightarrow$ L. Ind.

b). $(1,0,1)^T$, $(0,1,0)^T$

sol $c_1 v_1 + c_2 v_2 = \bar{0}$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{l} c_2 = 0 \\ c_1 = 0 \end{array}$$

unique solution. $\Rightarrow$ L.Ind.

$\boxed{c}$ $(1,2,4)^T$, $(2,1,3)^T$, $(4,-1,1)^T$

sol $c_1 v_1 + c_2 v_2 + c_3 v_3 = \bar{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 1 & -1 & 0 \\ 4 & 3 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -3 & -9 & 0 \\ 0 & -5 & -15 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & \; \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

c_3 is free variable, $c_3 = \alpha$

$\Rightarrow$ Infinite number of solution $\Rightarrow$ L.D.

ex Are $E_{11}, E_{12}, E_{21}, E_{22}$ L.Ind. in $\mathbb{R}^{2\times2}$

sol $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow c_1 = c_2 = c_3 = c_4 = 0$$

unique sol.

$\Rightarrow$ Lin. Independent.

Thm: Let $x_1, x_2, \dots, x_n$ be vectors in $\mathbb{R}^n$ and Let $\underset{n\times n}{X} = (x_1 \dots x_n)$ be a matrix. then $x_1, x_2, \dots, x_n$ are Linearly dep iff $\underset{n\times n}{X}$ is a singular matrix

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i.e $|\underset{n\times n}{X}| = 0$ if and only if $x_1, \dots, x_n$ Linearly dep.

$|\underset{n\times n}{X}| \neq 0 \Longleftrightarrow x_1, \dots, x_n$ Linearly Indep.

Ex Determine whether the vectors $(1,2,4)^T$, $(2,1,3)^T$, $(4,-1,1)^T$ are Linearly Independent or Linearly dependent.

Sol

$$X = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 4 & 3 & 1 \end{bmatrix} \Rightarrow |X| = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$$

$$= 4 - 2(6) + 4(2)$$

$$= 4 - 12 + 8 = 0$$

$\Rightarrow |X| = 0 \Rightarrow x_1, x_2, x_3$ are Linearly dependent.

H.W

use det.

1. Are $(4,2,3)^T$, $(2,3,1)^T$, $(2,-5,3)^T$ L.d. or L.Ind.
2. Are $(1,-1,2,3)^T$, $(-2,3,1,-2)^T$, $(1,0,7,7)^T$ L.d. or L.Ind.

Can not use deter.. use $c_1v_1 + \dots$

Note: let $x_1, x_2, \dots, x_n$ be

1. Linearly Independent set.
   - add $\Downarrow$ dep / Ind.
   - delete $\Downarrow$ Indep.

or

2. Linearly dependent set.
   - add $\Downarrow$ dep
   - delete: dep / Ind.

3.4 Basis And Dimension

Def The vectors $v_1, v_2, \dots, v_n$ form a basis for a vector space V if and only if:

(i) $v_1, \dots, v_n$ are Linearly independent.
(ii) $v_1, \dots, v_n$ span V.

Def. The dimension of a Vector space V is the number of vectors of a basis of V

ex Is the set $\{e_1, e_2, e_3\}$ form a basis for $\mathbb{R}^3$.

sol ① the vectors e_1, e_2, e_3 are Linearly Independent since

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0 .$$

② span $\{e_1, e_2, e_3\} = \mathbb{R}^3$

$\Rightarrow$ $\{e_1, e_2, e_3\}$ for a basis for $\mathbb{R}^3$

Dimension of $\mathbb{R}^3 = 3$

ex Is the set $\{(1,1,1), (1,1,0), (1,0,1)\}$ a basis for $\mathbb{R}^3$

sol ①

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow \text{Linearly independent.}$$

② $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = (a, b, c)$ span (v_1, v_2, v_3)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 1 & 0 & b \\ 1 & 0 & 1 & c \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 0 & -1 & b-a \\ 0 & -1 & 0 & c-a \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-c \\ 0 & 0 & 1 & a-b \end{array}\right] \end{array}$$

$\alpha_3 = a-b$
$\alpha_2 = a-c$
$\alpha_1 = \dots$

$\Rightarrow \exists\, \alpha_1, \alpha_2, \alpha_3 ; \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = (a, b, c)$

$\Rightarrow v_1, v_2, v_3$ span $\mathbb{R}^3$

$\Rightarrow \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

note dimension of $\mathbb{R}^3 = 3$

Note ① A basis of a Vector space is not unique.

② the dimension of a vector space is unique.

i.e let $\{v_1, \dots, v_n\}$ and $\{u_1, \dots, u_m\}$ be two basis for V, then $n = m$.

Standard basis for $\mathbb{R}^n$ is the set $\{e_1, e_2, \dots, e_n\}$

ex $\{e_1, e_2\}$ is a standard basis for $\mathbb{R}^2$

$\{e_1, e_2, e_3, e_4\}$ is a standard basis for $\mathbb{R}^4$

$\vdots$

ex show that $E_{11}, E_{12}, E_{21}, E_{22}$ is ~~a stan.~~ a basis for $\mathbb{R}^{2\times 2}$.

sol $\boxed{1}$ Are $E_{11}, E_{12}, E_{21}, E_{22}$ are Linearly Independent: Yes

because ..... done.

$\boxed{2}$ Is span $(E_{11}, E_{12}, E_{21}, E_{22}) = \mathbb{R}^{2\times 2}$: Yes

because $\alpha_1 E_{11} + \alpha_2 E_{12} + \alpha_3 E_{21} + \alpha_4 E_{22} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} \alpha_1 = a \\ \alpha_2 = b \\ \alpha_3 = c \\ \alpha_4 = d \end{matrix}$$

$\exists\, \alpha_1, \dots, \alpha_4$ $\{$

$\Rightarrow$

Note $\dim(\mathbb{R}^{2\times 2}) = 4$.

ex Is $S = \{(1,2,3)^T, (0,1,5)^T\}$ a basis for $\mathbb{R}^3$

sol no since S has just two vectors and

the $\dim(\mathbb{R}^3) = 3$

any basis of $\mathbb{R}^3$ must has 3 vectors.

Thm: If $\{v_1, \dots, v_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$ is Linearly depend

ex $\{e_1, e_2, e_3, e_4\}$ is a spanning set for $\mathbb{R}^4$

$\Rightarrow \{e_1, e_2, e_3, e_4, (2,0,1,3)^T\}$, $\{(2,1,0,0), (-1,5,4,3), (2,5,-2,4), (7,8,2,\sqrt{3}), (5,7,-1,1)\}$

L.D $\qquad$ L.D

Thm If V is a vector space of dimension $n > 0$

1. Any set of n Linearly independent Vectors span V.
2. Any n vectors that span V are Linearly independent.

ex show that $S = \{(1,2,3)^T, (-2,1,0), (1,0,1)\}$ is a basis for $\mathbb{R}^3$.

sol since S has 3 vectors and $\dim(\mathbb{R}^3) = 3$

1. $$\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 3(-1) + (1+4) = -3+5 = 2 \neq 0$$

$\Rightarrow v_1, v_2, v_3$ are Linearly Indep.

$\Rightarrow \text{span}(v_1, v_2, v_3) = \mathbb{R}^3$

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ونفحص L.Indep

Thm If V is a vector space of dimension $n > 0$ then No set of Less than n vectors can span V.
(so can not form a basis).

ex Is $S = \{(2,3,7), (-5,4,3)\}$ a spanning set for $\mathbb{R}^3$

sol No, because this set has just two elements and $\dim(\mathbb{R}^3) = 3$

note that S is not a basis for $\mathbb{R}^3$ (but v_1, v_2 may be L.Ind.)

Note! let V be vectorspace of dimension $n > 0$

take a set of vectors that contains

- $m < n$ elements $\Downarrow$ not span, not basis
- n elements $\Downarrow$ check: Indep. (أسهل) or span
- $m > n$ elem $\Downarrow$ not Linearly Indep. (L.D), not ~~span~~ basis

Is $S = \{v_1, v_2, v_3, v_4, v_5\}$ Linearly dep. in $\mathbb{R}^4$

sol No, since S has 5 vectors and $\dim(\mathbb{R}^4) = 4$

S is a L.D set.

ex Give an example of :

① 2 vectors form a basis for $\mathbb{R}^3$: No example (can not span)

② 4 vectors form a basis for $\mathbb{R}^3$: No example (can not be Lin. Ind)

③ 3 vectors : a basis for $\mathbb{R}^3$: $\{e_1, e_2, e_3\}$.

ex Give an example of 3 vectors span $\mathbb{R}^2$

solution $\{e_1, e_2, (2,3)\}$

Thm If V is a vector space of dimension $n > 0$, then

① Any subset of Less than n Linearly independent vectors can be extended to form a basis for V.

② Any spanning set of more than n vectors can be pared down to form a basis for V.

ex ① take the set $\{e_1, e_2\}$ is $\mathbb{R}^3$ L.Indep set with 2 elements < 3

extend the set to be $\{e_1, e_2, e_3\} \Rightarrow$ a basis for $\mathbb{R}^3$

② take the set $\{e_1, e_2, e_3, (1,0,3)^T\}$ span $\mathbb{R}^3$

pare down the set to be $\{e_1, e_2, e_3\} \Rightarrow$ a basis for $\mathbb{R}^3$.

ex take $x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $x_4 = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$, $x_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

① $\text{Span}(x_1, \dots, x_5) = \mathbb{R}^3$ ...

② Pare down x_3, x_4, then the set $\{x_1, x_2, x_5\}$ is a basis for $\mathbb{R}^3$

since $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 2 & 4 & 0 \end{vmatrix} = 1(8-10) - 1(4-4) = -2 \neq 0$