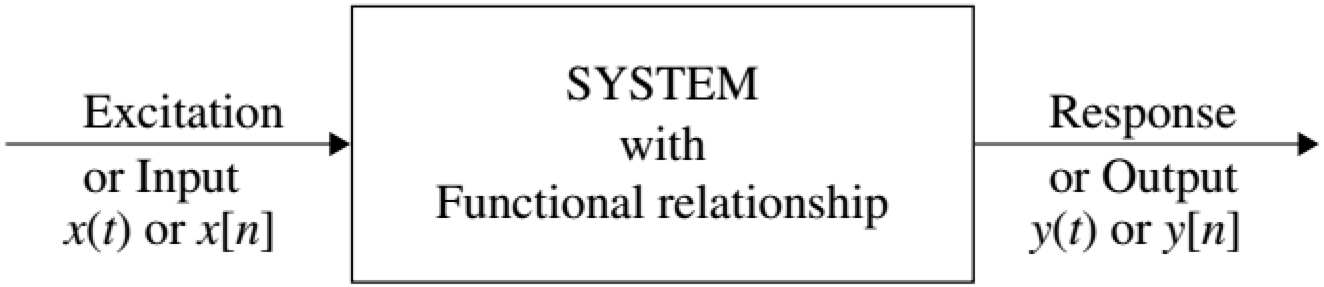
Signals and systems

Chapter 2: systems

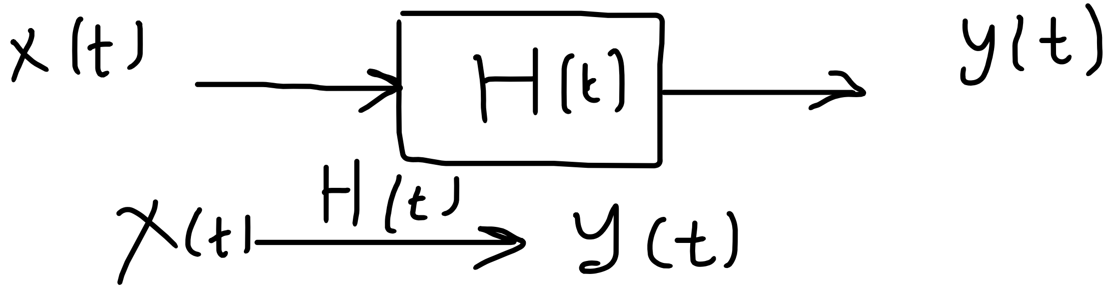
Introduction:

- The system is any process that results in transformation on a signal
- It has an input
- An out put
- A transformation function

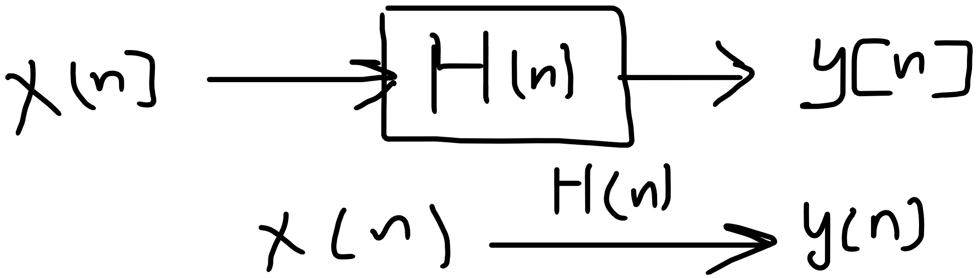


Systems has two types :

- 1-Linear Time Invariant Continuous (LTIC) Time System



- 2- Linear Time Invariant Discrete (LTID) Time System



Properties (Classification) of Continuous Time System

Linear and Non-linear Systems

For a system

if $x_1(t) \longrightarrow y_1(t)$ and $x_2(t) \longrightarrow y_2(t)$

The system is linear if

$$[x_1(t) + x_2(t)] \longrightarrow y(t) = y_1(t) + y_2(t)$$

- For composite signal

$$a\,x_1(t) \longrightarrow a\,y_1(t)$$

$$b\,x_2(t) \longrightarrow a\,y_2(t)$$

$$\left[a\,x_1(t) + b\,x_2(t)\right] \longrightarrow a\,y_1(t) + b\,y_2(t)$$

- Thus, for a continuous system to be linear, the weighted sum of several inputs produces the weighted sum of outputs. In other words, it should satisfy the homogeneity and additivity properties of superposition theorem. If the above conditions are not satisfied the system is said to be non-linear.

Step By Step Procedure to Test Linearity

1. Let

$$y_1(t) = f(x_1(t))$$
$$y_2(t) = f(x_2(t))$$

Find the weighted sum of the output

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$$
$$y_3(t) = a_1 f(x_1(t)) + a_2 f(x_2(t))$$

where a_1 and a_2 are called the weights.

2. For the linear combination of input $[a_1x_1(t) + a_2x_2(t)]$ find the output for the weighted sum of the input.

$$y_4(t) = f[a_1x_1(t) + a_2x_2(t)]$$

3. If

$$y_3(t) = y_4(t)$$

the system is linear. Otherwise the system is non-linear. The following examples, illustrate the method of testing the linearity of continuous time systems.

4. If the output is not zero for zero input, the system will be non-linear.

Consider the following input-output equation of a certain system.

$$y(t) = [2x(t)]^2$$

Determine whether the system is linear or non-linear.

Solution:

$$\begin{aligned} y(t) &= [2x(t)]^2 \\ &= 4x^2(t) \\ y_1(t) &= 4x_1^2(t) \\ y_2(t) &= 4x_2^2(t) \end{aligned}$$

The weighted sum of the output is,

$$\begin{aligned} y_3(t) &= a_1 y_1(t) + a_2 y_2(t) \\ &= 4a_1 x_1^2(t) + 4a_2 x_2^2(t) \end{aligned}$$

The output due to the weighted sum of the input $[a_1 x_1 + a_2 x_2]$ is,

$$\begin{aligned} y_4(t) &= 4[a_1 x_1(t) + a_2 x_2(t)]^2 \\ &= 4[a_1^2 x_1^2(t) + a_2^2 x_2^2(t) + 2a_1 a_2 x_1(t) x_2(t)] \\ y_3(t) &\neq y_4(t) \end{aligned}$$

Hence, the system is non-linear.

$$\boldsymbol{y(t) = t^2 x(t+1)}$$

$$y_1(t) = t^2 x_1(t+1)$$

$$y_2(t) = t^2 x_2(t+1)$$

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$$

$$= t^2[a_1 x_1(t+1) + a_2 x_2(t+1)]$$

$$y_4(t) = t^2[a_1 x_1(t+1) + a_2 x_2(t+1)]$$

$$y_3(t) = y_4(t)$$

The system is Linear.

$\boldsymbol{y(t) = E_v x(t)}$

$$y(t) = \frac{1}{2}[x(t) + x(-t)]$$
$$y_1(t) = \frac{1}{2}[x_1(t) + x_1(-t)]$$
$$y_2(t) = \frac{1}{2}[x_2(t) + x_2(-t)]$$

The weighted sum of the output is,

$$\begin{aligned} y_3(t) &= a_1 y_1(t) + a_2 y_2(t) \\ &= \frac{1}{2}[a_1 x_1(t) + a_2 x_2(t) + a_1 x_1(-t) + a_2 x_2(-t)] \end{aligned}$$

The output due to the weighted sum of the input is,

$$\begin{aligned} y_4(t) &= \frac{1}{2}[a_1(x_1(t) + x_1(-t)) + a_2(x_2(t) + x_2(-t))] \\ &= \frac{1}{2}[a_1 x_1(t) + a_2 x_2(t) + a_1 x_1(-t) + a_2 x_2(-t)] \\ y_3(t) &= y_4(t) \end{aligned}$$

The system is Linear.

$$\boldsymbol{y(t) = e^{-2x(t)}}$$

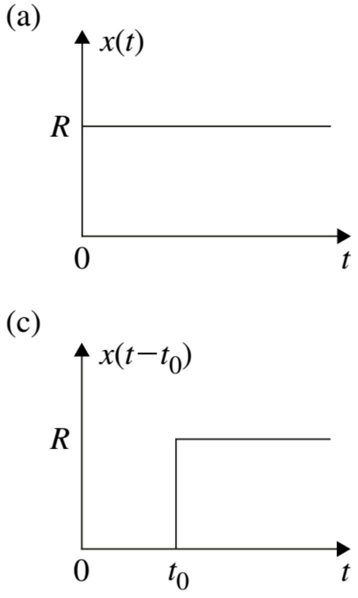
For $x(t) = 0, y(t) = 1$ and not zero. Hence the system is non-linear. Also

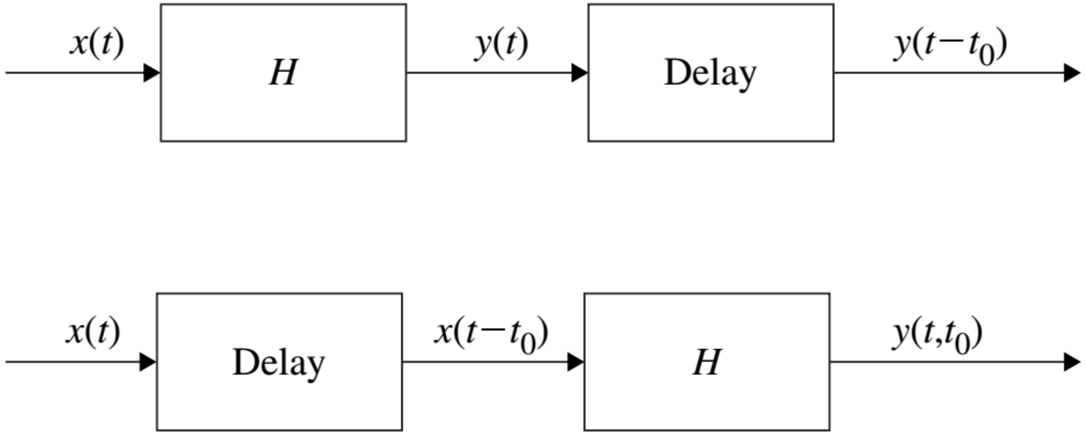
$$y_1(t) = e^{-2x_1(t)}$$
$$y_2(t) = e^{-2x_2(t)}$$
$$y_3(t) = a_1 y_1(t) + a_2 y_2(t) = a_1 e^{-2x_1(t)} + a_2 e^{-2x_2(t)}$$
$$y_4(t) = e^{-2(a_1 x_1(t) + a_2 x_2(t))} = e^{-2a_1 x_1(t)} e^{-2a_2 x_2(t)}$$
$$y_3(t) \neq y_4(t)$$

The system is Non-linear.

Time Invariant and Time Varying Systems

- A continuous time system is said to be time invariant if the parameters of the system do not change with time





Step 1. For the delayed input $x(t - t_0)$ obtain the output $y(t, t_0)$.

Step 2. Obtain the expression for the delayed output $y(t - t_0)$ by substituting $t = (t - t_0)$.

Step 3. If $y(t, t_0) = y(t - t_0)$, then the system is time invariant. Otherwise it is a time varying system.

$\boldsymbol{y(t) = tx(t)}$

1. For the delayed input $x(t - t_0)$, the output $y(t, t_0)$ is obtained as

$$y(t, t_0) = tx(t - t_0)$$

2. The delayed output $y(t - t_0)$ is obtained by substituting $t = t - t_0$ in the given equation

$$y(t - t_0) = (t - t_0)x(t - t_0)$$

3. $y(t - t_0) \neq y(t, t_0)$
4.

The system is Time Varying.

$$\boldsymbol{y(t) = \cos x(t)}$$

1. $y(t, t_0) = \cos x(t - t_0)$ [For Delayed input]
2. $y(t - t_0) = \cos x(t - t_0)$ [Delayed output]
3. $y(t - t_0) = y(t, t_0)$
4.

The system is Time Invariant.

$$\boldsymbol{y(t) = e^{-2x(t)}}$$

1. The output due to delayed input is,

$$y(t, t_0) = e^{-2x(t-t_0)}$$

2. The delayed output is obtained by putting $t = t - t_0$

$$y(t - t_0) = e^{-2x(t-t_0)}$$

3. $y(t - t_0) = y(t, t_0)$
4.

The system is Time Invariant.

$$\boldsymbol{y(t) = at^2x(t) + btx(t-2)}$$

The output $y(t, t_0)$ due to the delayed input $x(t - t_0)$ is

$$y(t, t_0) = at^2x(t - t_0) + btx(t - t_0 - 2)$$

The delayed output $y(t - t_0)$ is obtained by substituting $t = t - t_0$.

$$y(t - t_0) = a(t - t_0)^2x(t - t_0) + b(t - t_0)x(t - t_0 - 2)$$

From equations (a) and (b) we see

$$y(t, t_0) \neq y(t - t_0)$$

The system is Time Varying.

Static and Dynamic Systems (Memoryless and System with Memory)

- A dynamic system is defined as a system in which the output signal at any specified time depends on the values of the input signals at the specific time at other time also.
- A static system is defined as a system in which the output signal at any specified time depends on the present value of the input signal alone. Static system is also called as instantaneous system

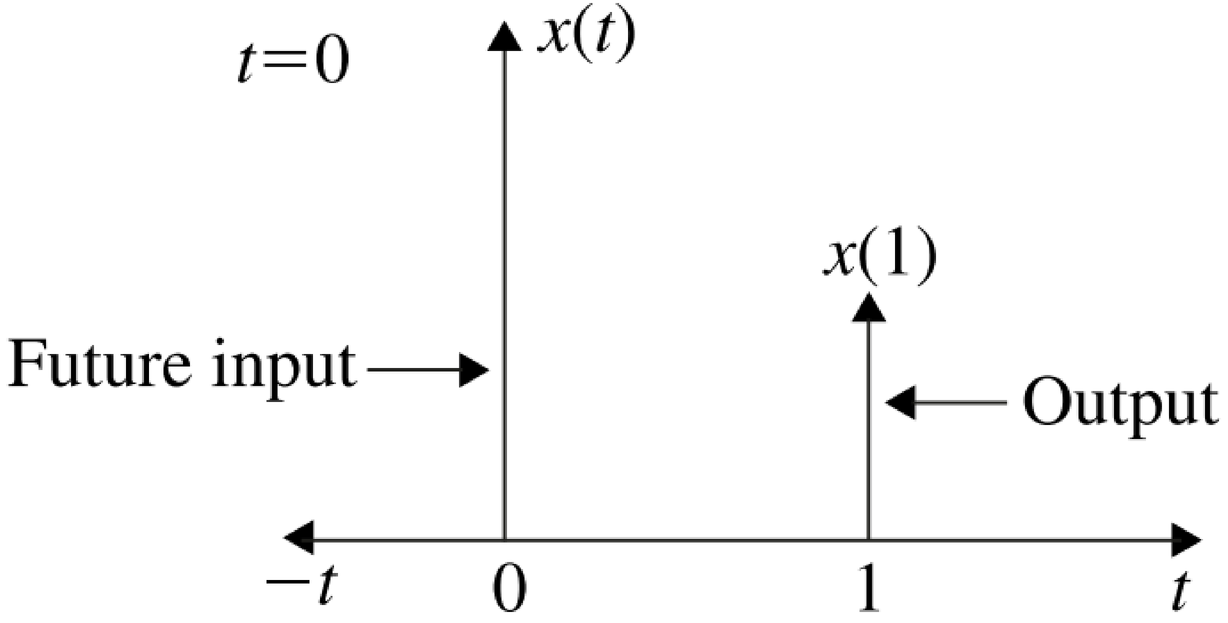
$$t = t_0, \text{ Present input.}$$

$$t < t_0, \text{ Past input.}$$

$$t > t_0, \text{ Future input.}$$

(a) $\boldsymbol{y(t) = x(t+1) + 5}$

$$y(0) = x(1) + 5$$



The system response depends on the future input $x(t+1)$ where $t > t_0$. Hence

The system is Dynamic.

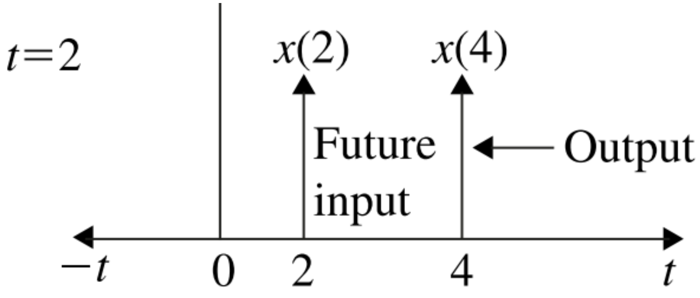
(b) $\boldsymbol{y(t) = x(t^2)}$

For $t = 1$,

$$y(1) = x(1) \quad [t = t_0 \text{ Present input}]$$

For $t = 2$,

$$y(2) = x(4)$$



The response depends on the present and future inputs. The output $x(4)$ depends upon the future input $x(2)$. Hence

The system is Dynamic.

(h) $\boldsymbol{y(t) = 2x(t) + 3}$

The output always depends on the present input. Hence

The system is Static.

(i) $\boldsymbol{y(t) = e^{-2x(t)}}$

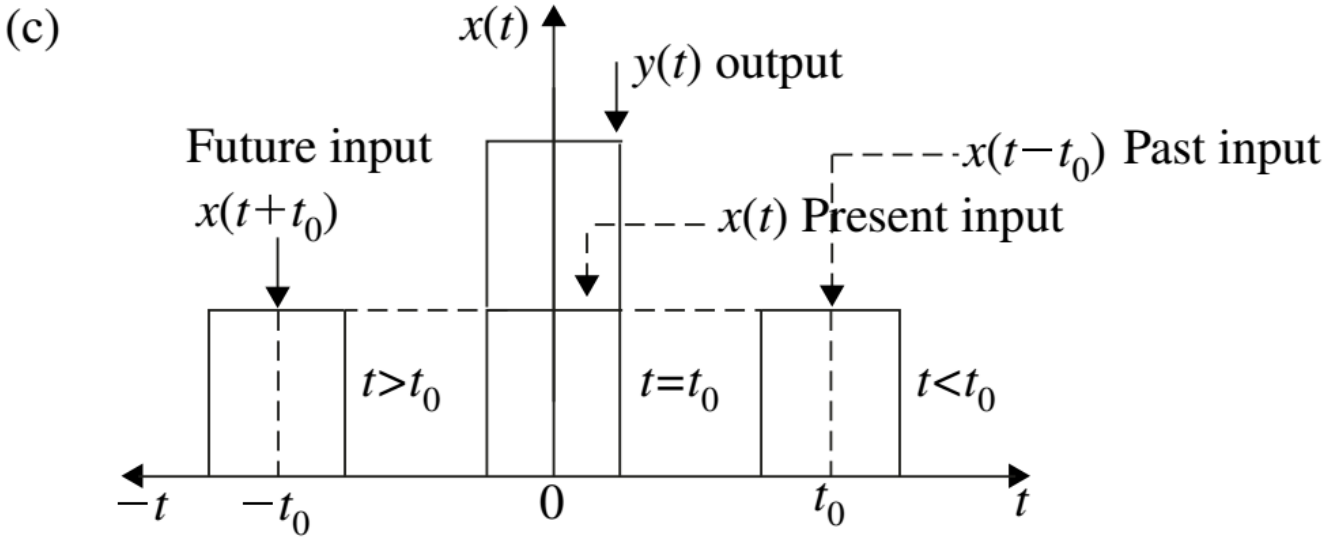
The output always depends on the present input only. Hence

The system is Static.

Causal and Non-causal Systems

- If the response (output) depends on the present and past values of the input x(t), the system is said to be causal.

$$y(t) = x(t-3) + x(t+3)$$



$$y(t) = x\left(\frac{t}{4}\right)$$

$$\text{For } t = 0, \quad y(0) = x(0)$$

$$\text{For } t = -4, \quad y(-4) = x(-1)$$

$$\text{For } t = 1, \quad y(1) = x\left(\frac{1}{4}\right)$$

from $y(-4) = x(-1)$.

The system is Non-causal.

$$\boldsymbol{y(t) = x(t)\sin(1+t)}$$

$$y(0) = x(0)\sin(1)$$
$$y(1) = x(1)\sin(2)$$
$$y(-1) = x(-1)\sin(0)$$

Thus at all time, the output depends on the present input only. Hence

$$\boxed{\text{The system is Causal.}}$$

$$\boldsymbol{y(t) = x(t^2)}$$

$$\text{For } t = 0, \quad y(0) = x(0)$$

$$\text{For } t = 1, \quad y(1) = x(1)$$

$$\text{For } t = 2, \quad y(2) = x(4)$$

The system output $y(t)$ at $t = 2$, which is $y(2) = x(4)$ depends on the future input $x(t)$. Hence

The system is Non-causal.

$$y(t) = x(t-1)$$

For $t = 0, \quad y(0) = x(-1)$

For $t = 1, \quad y(1) = x(0)$

For $t = 2, \quad y(2) = x(1)$

The output depends on the past values of the input.

The system is Causal.

Stable and Unstable Systems

- A linear time invariant continuous time system is said to be Bounded Input Bounded Output (BIBO) stable, if for any bounded input, it produces bounded output. This also implies that for BIBO stability, the area under the impulse response (output) curve should be finite.

$$y(t) = f[x(t)] \qquad \text{for all } t \tag{2.8}$$

If $|x(t)|$ is bounded, $|y(t)|$ should also be bounded for the system to be stable.

$$|y(t)| \leq M_y < \infty \qquad \text{for all } t \tag{2.9}$$

$$|x(t)| \leq M_x < \infty \qquad \text{for all } t \tag{2.10}$$

where $|M_x|$ and $|M_y|$ represent positive values. It can be easily established that the necessary and sufficient condition for the LTIC time system to be stable is,

$$\boxed{y(t) = \int_{-\infty}^{\infty} |x(t)| dt < \infty}$$

$\boldsymbol{y(t) = tx(t)}$

If $x(t)$ is bounded, $y(t)$ varies with respect to time and becomes unbounded. Hence

The system is BIBO Unstable.

$\boldsymbol{y(t) = x(t)\sin t}$

It $x(t)$ is bounded, $y(t)$ is also bounded because $\sin t$ will take a maximum value of $+1$ and -1. Hence, $y(t)$ is bounded.

The system is BIBO Stable.

$\boldsymbol{y(t) = e^{-2|t|}}$

Here

$$
\begin{aligned}
x(t) &= e^{-2t} \qquad\qquad 0 \le t < \infty \\
&= e^{2t} \qquad\qquad -\infty < t < 0 \\
y(t) &= \int_{-\infty}^{\infty} x(t)dt \\
&= \int_{-\infty}^{0} e^{2t}dt + \int_{0}^{\infty} e^{-2t}dt \\
&= \left[\frac{1}{2}e^{2t}\right]_{-\infty}^{0} - \left[\frac{1}{2}e^{-2t}\right]_{0}^{\infty} \\
&= \frac{1}{2}[1 + 1] = 1 < \infty
\end{aligned}
$$

The output is bounded and the system is stable.

The system is BIBO Stable.

$\boldsymbol{y(t) = te^{2t}u(t)}$

Here the output varies linearly as t and also exponentially increasing due to e^{2t}. Hence, $|y(t)| = \infty$ and the system is BIBO unstable. Mathematically this can be proved as follows. For a causal system, $|y(t)|$ can be written as

$$|y(t)| = \int_0^\infty te^{2t}dt$$

The following integration formula is used to evaluate the above integral.

$$\int_0^\infty te^{at}dt = \frac{1}{a^2}\Big[e^{at}\{at-1\}\Big]_0^\infty$$

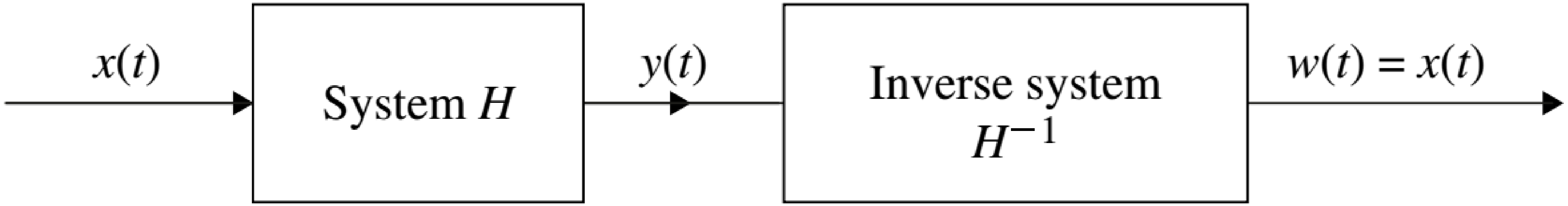
$$\begin{aligned}|y(t)| &= \frac{1}{4}\Big[e^{2t}\{2t-1\}\Big]_0^\infty \\ &= \frac{1}{4}[e^\infty\{2\infty - 1\} + 1] \\ &= \infty\end{aligned}$$

The system is BIBO Unstable.

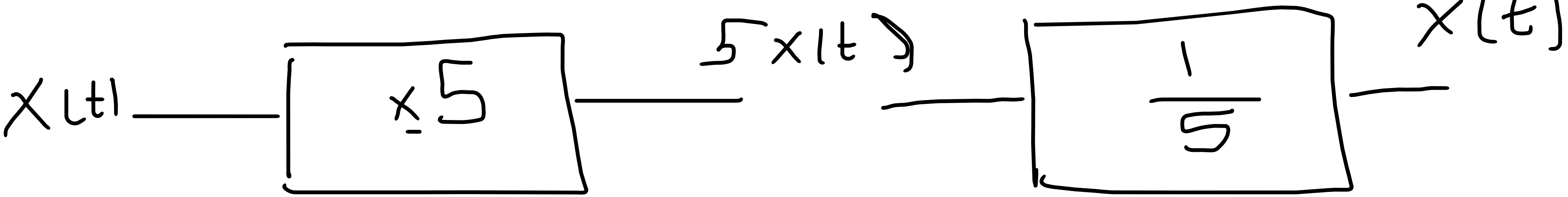
Invertibility and Inverse System

A system is said to be invertible if the distinct inputs give distinct output.

(a)



$y(t) = 5x(t)$



inversable

$y(t) = x^2(t)$ non inversable

Discrete Time System

Linear and Non-linear Systems

$$a_1x_1[n] + a_2x_2[n] \Rightarrow a_1y_1[n] + a_2y_2[n]$$

$$[a_1 x_1 + a_2 x_2] \Rightarrow y = a_1 y_1 + a_2 y_2$$

$\boldsymbol{y[n] = x^2[n]}$

$$y_1[n] = x_1^2[n]$$
$$y_2[n] = x_2^2[n]$$

1. The weighted sum of the output $y_3[n]$ is,

$$\begin{aligned} y_3[n] &= a_1 y_1[n] + a_2 y_2[n] \\ &= a_1 x_1^2[n] + a_2 x_2^2[n] \end{aligned}$$

2. The output $y_4[n]$ due to the weighted sum of the input is,

$$\begin{aligned} y_4[n] &= [a_1 x_1[n] + a_2 x_2[n]]^2 \\ &= a_1^2 x_1^2[n] + a_2^2 x_2^2[n] + 2a_1 a_2 x_1[n] x_2[n] \end{aligned}$$

3.

$$y_3[n] \neq y_4[n]$$

The system is Non-linear.

$y[n] = x[4n+1]$

$$a_1 y_1[n] = a_1 x_1[4n+1]$$
$$a_2 y_2[n] = a_2 x_2[4n+1]$$
$$y_3[n] = a_1 y_1[n] + a_2 y_2[n]$$

1. The weighted sum of the output is,

$$\begin{aligned} y_3[n] &= a_1 y_1[n] + a_2 y_2[n] \\ &= a_1 x_1[4n+1] + a_2 x_2[4n+1] \end{aligned}$$

2. The output due to the weighted sum of the input is,

$$y_4[n] = a_1 x_1[4n+1] + a_2 x_2[4n+1]$$

3.

$$y_3[n] = y_4[n]$$

The system is Linear.

$y[n] = x[n^2]$

$$a_1 y_1[n] = a_1 x_1[n^2]$$
$$a_2 y_2[n] = a_2 x_2[n^2]$$

1. The weighted sum of the output $y_3[n]$ is,

$$\begin{aligned} y_3[n] &= a_1 y_1[n] + a_2 y_2[n] \\ &= a_1 x_1[n^2] + a_2 x_2[n^2] \end{aligned}$$

2. The output $y_4[n]$ due to the weighted sum of input is,

$$y_4[n] = a_1 x_1[n^2] + a_2 x_2[n^2]$$

3.

$$y_3[n] = y_4[n]$$

The system is Linear.

Time Invariant and Time Varying DT Systems

1. For the delayed input $x[n - n_0]$ find the output $y[n, n_0]$.
2. Obtain the delayed output $y[n - n_0]$ by substituting $n = n - n_0$ in $y[n]$.
3. If $y[n, n_0] = y[n - n_0]$, the system is time invariant. Otherwise the system is time varying.

$\boldsymbol{y[n] = nx[n]}$

1. The output for the delayed input $x[n-n_0]$ is obtained by delaying the input $x[n]$ as $x[n-n_0]$. Thus

$$y[n, n_0] = nx[n-n_0]$$

2. The delayed output for the input $x[n]$ is obtained by substituting $n = n - n_0$.

$$y[n-n_0] = (n-n_0)x[n-n_0]$$

3.

$$y[n, n_0] \neq y[n-n_0]$$

The system is Time Variant.

$$\boldsymbol{y[n] = \sin(x[n])}$$

The output due to delayed input is

$$y[n, n_0] = \sin(x[n - n_0])$$

The delayed output is,

$$y[n - n_0] = \sin(x[n - n_0])$$
$$y[n, n_0] = y[n - n_0]$$

The system is Time Invariant.

$$\boldsymbol{y[n] = x[n]x[n-1]}$$

The output due to delayed input is

$$y[n, n_0] = x[n - n_0]x[n - n_0 - 1]$$

The delayed output is,

$$y[n - n_0] = x[n - n_0]x[n - n_0 - 1]$$

$$y[n, n_0] = y[n - n_0]$$

The system is Time Invariant.

Causal and Non-causal DT Systems

- A discrete time system is said to be causal if the response of the system depends on the present or the past inputs applied. The systems is non-causal if the output depends on the future input.

$$\boldsymbol{y[n] = x[n-1]}$$

$$y[0] = x[-1]$$
$$y[1] = x[0]$$

$x[n-1]$ is the past input for the output $y[n]$. The output depends on the past value of $x[n]$. Hence

The system is Causal.

$$\boldsymbol{y[n] = x[n] + x[n-1]}$$

$$\text{For } n = 0, \quad y[0] = x[0] + x[-1]$$
$$\text{For } n = 1, \quad y[1] = x[1] + x[0]$$

Here $x[n]$ is present value and $x[n-1]$ is past value. The output depends on the present and past inputs. Hence

The system is Causal.

$$\boldsymbol{y[n-1] = x[n]}$$

Put $n = n + 1$

$$y[n] = x[n+1]$$

$$y[0] = x[1]$$

The output depends on the future inputs. Hence

The system is Non-causal.

$$y[n] = \sum_{k=-\infty}^{n+4} x[k]$$

$$y[0] = \sum_{-\infty}^{4} x[k]$$

$$= x[-\infty]+x[-\infty + 1]+ \cdots +x[-1]+x[0]+x[1]+x[2]+x[3]+x[4]$$

$x[-\infty] + x[-\infty + 1], \ldots, x[-1]$ = Future output for past input

$x[0]$ = Present output for present input

$x[1],\ x[2],\ x[3]$ and $x[4]$ = Past output for future input

The system is Non-causal.

$$\boldsymbol{y[n] = \sum_{k=0}^{n-3} x[k]}$$

$$y[0] = \sum_{k=0}^{-3} x[k]$$
$$= x[0] + x[-1] + x[-2] + x[-3]$$
$$x[0] = \text{Present output for present input}$$
$$x[-1], x[-2], x[-3] = \text{Future outputs for past input}$$

The output depends on the present and past inputs. Hence

$$\boxed{\text{The system is Causal.}}$$

Stable and Unstable Systems

A discrete time system is said to be stable if for any bounded input, it produces a bounded output. This implies that the impulse response

$$y[n] = \sum_{-\infty}^{\infty} |h[n]| < \infty$$

is absolutely summable.

$\boldsymbol{y[n] = \sin x[n]}$ If $x[n]$ is bounded, then $\sin x[n]$ is also bounded and so $y[n]$ is also bounded

The system is Stable.

$\boldsymbol{y[n] = \sum_{k=0}^{n+1} x[k]}$

Here as $n \to \infty$, $y[n] \to \infty$ and the output is unbounded. For bounded input n should be a finite number.

In that case $y[n]$ is bounded and the system is stable.

The system is Stable. for $n = \text{finite}$

The system is Unstable. for $n = \infty$

$$\boldsymbol{h[n] = 3^n u[n+3]}$$

$$|y[n]| = \sum_{n=-3}^{\infty} 3^n$$
$$= (3)^{-3} + (3)^{-2} + (3)^{-1} + (3)^0 + (3)^1 + \cdots + (3)^{\infty}$$
$$= \infty$$

The output is unbounded.

The system is Unstable.

$$y[n] = \delta[n-1] + \delta[n] + \delta[n+1]$$

$$y[0] = \delta[-1] + \delta[0] + \delta[1] = 0 + 1 + 0 = 1$$
$$y[1] = \delta[0] + \delta[1] + \delta[2] = 1 + 0 + 0 = 1$$
$$y[-1] = \delta[-2] + \delta[-1] + \delta[0] = 0 + 0 + 1 = 1$$
$$y[-2] = \delta[1] + \delta[2] + \delta[3] = 0 + 0 + 0 = 0$$
$$y[2] = \delta[1] + \delta[2] + \delta[3] = 0 + 0 + 0 = 0$$
$$y[n] = \sum_{-\infty}^{\infty} |h[k]| = 1 + 1 + 1 = 3 < \infty$$

The system is Stable.

$$\boldsymbol{h[n] = n\,u[n]}$$

$$y[n] = \sum_{0}^{\infty} n = 1 + 2 + \cdots + \infty = \infty$$

The system is Unstable.

Static and Dynamic Systems

- A discrete time system is said to be static (memoryless or instantaneous) if the output response depends on the present value only and not on the past and future values of excitation. Discrete systems described by difference equations require memory and hence they are dynamic systems.

$$y[n] = x[3n]$$

$$\text{For } n = 0, \quad y[0] = x[0]$$
$$\text{For } n = 1, \quad y[1] = x[3]$$
$$\text{For } n = -1, \quad y[-1] = x[-3]$$

The outputs $y[0] = x[0]$, $y[1] = x[3]$ and $y[-1] = x[-3]$ depend upon the present input, future input and past input respectively.

The system is Dynamic.

$$\boldsymbol{y[n] = \sin(x[n])}$$

$$y[0] = \sin(x[0])$$

$$y[1] = \sin(x[1])$$

The output depends on the present input at all time. Hence

The system is Static.

Invertible and Inverse Discrete Time Systems

- A discrete time system is said to be invertible if distinct input leads to distinct output. If a system is invertible then an inverse system exists