EXAMPLE 7.4 **Fourth-order system**

1. (a) We desire to plot the root locus for the characteristic equation of a system as K varies for $K > 0$ when

$$1 + \frac{K}{s^4 + 12s^3 + 64s^2 + 128s} = 0.$$

(b) Determining the poles, we have

$$1 + \frac{K}{s(s + 4)(s + 4 + j4)(s + 4 - j4)} = 0 \qquad (7.49)$$

as K varies from zero to infinity. This system has no finite zeros.

(c) The poles are located on the s-plane as shown in Figure 7.14(a).

(d) Because the number of poles n is equal to 4, we have four separate loci.

(e) The root loci are symmetrical with respect to the real axis.

2. A segment of the root locus exists on the real axis between $s = 0$ and $s = -4$.

3. The angles of the asymptotes are

$$\phi_A = \frac{(2k + 1)}{4}180°, \qquad k = 0, 1, 2, 3;$$

$$\phi_A = +45°, 135°, 225°, 315°.$$

The center of the asymptotes is

$$\sigma_A = \frac{-4 - 4 - 4}{4} = -3.$$

Then the asymptotes are drawn as shown in Figure 7.14(a).

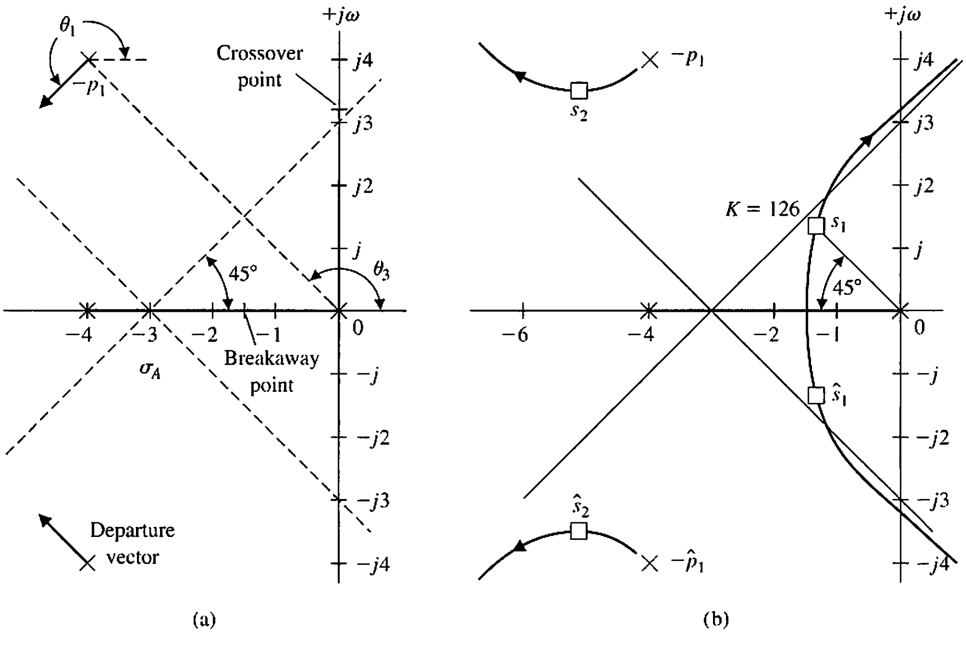

FIGURE 7.14 The root locus for Example 7.4. Locating (a) the poles and (b) the asymptotes.

4. The characteristic equation is rewritten as

$$s(s + 4)(s^2 + 8s + 32) + K = s^4 + 12s^3 + 64s^2 + 128s + K = 0. \quad (7.50)$$

Therefore, the Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 64 & K \\ s^3 & 12 & 128 & \\ s^2 & b_1 & K & \\ s^1 & c_1 & & \\ s^0 & K & & \end{array},$$

where

$$b_1 = \frac{12(64) - 128}{12} = 53.33 \quad \text{and} \quad c_1 = \frac{53.33(128) - 12K}{53.33}.$$

Hence, the limiting value of gain for stability is $K = 568.89$, and the roots of the auxiliary equation are

$$53.33s^2 + 568.89 = 53.33(s^2 + 10.67) = 53.33(s + j3.266)(s - j3.266). \quad (7.51)$$

The points where the locus crosses the imaginary axis are shown in Figure 7.14(a). Therefore, when $K = 568.89$, the root locus crosses the $j\omega$-axis at $s = \pm j3.266$.

5. The breakaway point is estimated by evaluating

$$K = p(s) = -s(s + 4)(s + 4 + j4)(s + 4 - j4)$$

between $s = -4$ and $s = 0$. We expect the breakaway point to lie between $s = -3$ and $s = -1$, so we search for a maximum value of $p(s)$ in that region. The resulting values of $p(s)$ for several values of s are given in Table 7.3. The maximum of $p(s)$ is found to lie at approximately $s = -1.577$, as indicated in the table. A more accurate estimate of the breakaway point is normally not necessary. The breakaway point is then indicated on Figure 7.14(a).

6. The angle of departure at the complex pole p_1 can be estimated by utilizing the angle criterion as follows:

$$\theta_1 + 90^\circ + 90^\circ + \theta_3 = 180^\circ + k360^\circ.$$

Here, θ_3 is the angle subtended by the vector from pole p_3. The angles from the pole at $s = -4$ and $s = -4 - j4$ are each equal to 90°. Since $\theta_3 = 135°$, we find that

$$\theta_1 = -135^\circ \equiv +225^\circ,$$

as shown in Figure 7.14(a).

7. Complete the sketch as shown in Figure 7.14(b).

Table 7.3

$p(s)$	0	51.0	68.44	80.0	83.57	75.0	0
s	−4.0	−3.0	−2.5	−2.0	−1.577	−1.0	0