10.3 CASCADE COMPENSATION NETWORKS

In this section, we will consider the design of a cascade or feedback network, as shown in Figures 10.1(a) and (b), respectively. The compensation network function $G_c(s)$ is cascaded with the specified process $G(s)$ in order to provide a suitable loop transfer function $L(s) = G_c(s)G(s)H(s)$. The compensator $G_c(s)$ can be chosen to alter either the shape of the root locus or the frequency response. In either case, the network may be chosen to have a transfer function

$$G_c(s) = \frac{K\prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)}. \tag{10.1}$$

Then the problem reduces to the judicious selection of the poles and zeros of the compensator. To illustrate the properties of the compensation network, we will consider a first-order compensator. The compensation approach developed on the basis of a first-order compensator can then be extended to higher-order compensators, for example, by cascading several first-order compensators.

A compensator $G_c(s)$ is used with a process $G(s)$ so that the overall loop gain can be set to satisfy the steady-state error requirement, and then $G_c(s)$ is used to adjust the system dynamics favorably without affecting the steady-state error.

Consider the first-order compensator with the transfer function

$$\boxed{G_c(s) = \frac{K(s + z)}{s + p}.} \tag{10.2}$$

The design problem then becomes the selection of z, p, and K in order to provide a suitable performance. When $|z| < |p|$, the network is called a **phase-lead network** and has a pole–zero s-plane configuration, as shown in Figure 10.2. If the pole was negligible, that is, $|p| \gg |z|$, and the zero occurred at the origin of the s-plane, we would have a differentiator so that

$$G_c(s) \approx \frac{K}{p}s. \tag{10.3}$$

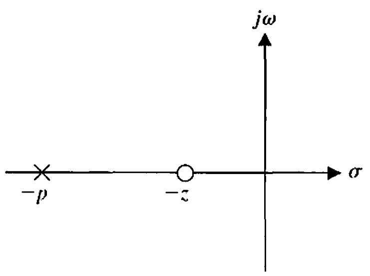


FIGURE 10.2 Pole–zero diagram of the phase-lead network.

Thus, a compensation network of the form of Equation (10.2) is a differentiator-type network. The differentiator network of Equation (10.3) has the frequency characteristic

$$G_c(j\omega) = j\frac{K}{p}\omega = \left(\frac{K}{p}\omega\right)e^{+j90^\circ} \tag{10.4}$$

and a phase angle of +90°. Similarly, the frequency response of the differentiating network of Equation (10.2) is

$$G_c(j\omega) = \frac{K(j\omega + z)}{j\omega + p} = \frac{(Kz/p)[j(\omega/z) + 1]}{j(\omega/p) + 1} = \frac{K_1(1 + j\omega\alpha\tau)}{1 + j\omega\tau}, \tag{10.5}$$

where $\tau = 1/p$, $p = \alpha z$, and $K_1 = K/\alpha$. The frequency response of this phase-lead network is shown in Figure 10.3. The angle of the frequency characteristic is

$$\phi(\omega) = \tan^{-1}(\alpha\omega\tau) - \tan^{-1}(\omega\tau). \tag{10.6}$$

Because the zero occurs first on the frequency axis, we obtain a phase-lead characteristic, as shown in Figure 10.3. The slope of the asymptotic magnitude curve is +20 dB/decade.

The phase-lead compensation transfer function can be obtained with the network shown in Figure 10.4. The transfer function of this network is

$$G_c(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2}{R_2 + \dfrac{R_1/(Cs)}{R_1 + 1/(Cs)}}$$

$$= \frac{R_2}{R_1 + R_2}\frac{R_1Cs + 1}{[R_1R_2/(R_1 + R_2)]Cs + 1}. \tag{10.7}$$

Therefore, we let

$$\tau = \frac{R_1R_2}{R_1 + R_2}C \quad \text{and} \quad \alpha = \frac{R_1 + R_2}{R_2},$$

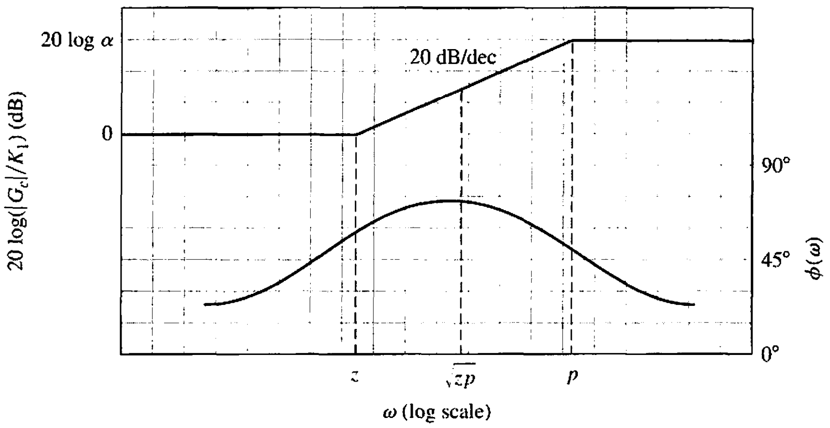


FIGURE 10.3 Bode diagram of the phase-lead network.

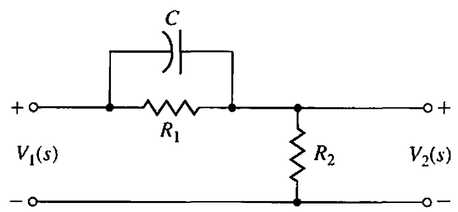


FIGURE 10.4 Phase-lead network.

and we obtain the **phase-lead compensation** transfer function

$$G_c(s) = \frac{1 + \alpha\tau s}{\alpha(1 + \tau s)}, \tag{10.8}$$

which is equal to Equation (10.5) when an additional cascade gain K is inserted.

The maximum value of the phase lead occurs at a frequency ω_m, where ω_m is the geometric mean of $p = 1/\tau$ and $z = 1/(\alpha\tau)$; that is, the maximum phase lead occurs halfway between the pole and zero frequencies on the logarithmic frequency scale. Therefore,

$$\omega_m = \sqrt{zp} = \frac{1}{\tau\sqrt{\alpha}}.$$

To obtain an equation for the maximum phase-lead angle, we rewrite the phase angle of Equation (10.5) as

$$\phi = \tan^{-1}\frac{\alpha\omega\tau - \omega\tau}{1 + (\omega\tau)^2\alpha}. \tag{10.9}$$

Then, substituting the frequency for the maximum phase angle, $\omega_m = 1/(\tau\sqrt{a})$, we have

$$\tan\phi_m = \frac{\alpha/\sqrt{\alpha} - 1/\sqrt{\alpha}}{1 + 1} = \frac{\alpha - 1}{2\sqrt{\alpha}}. \tag{10.10}$$

We use the trigonometric relationship $\sin\phi = \tan\phi/\sqrt{1 + \tan^2\phi}$ and obtain

$$\sin\phi_m = \frac{\alpha - 1}{\alpha + 1}. \tag{10.11}$$

Equation (10.11) is very useful for calculating a necessary α ratio between the pole and zero of a compensator in order to provide a required maximum phase lead. A plot of ϕ_m versus α is shown in Figure 10.5. The phase angle readily obtainable from this network is not much greater than 70°. Also, since $\alpha = (R_1 + R_2)/R_2$, there are practical limitations on the maximum value of α that we should attempt to obtain. Therefore, if we required a maximum angle greater than 70°, two cascade compensation networks would be used. Then the equivalent compensation transfer function would be $G_{c_1}(s)G_{c_2}(s)$ when the loading effect of $G_{c_2}(s)$ on $G_{c_1}(s)$ is negligible.

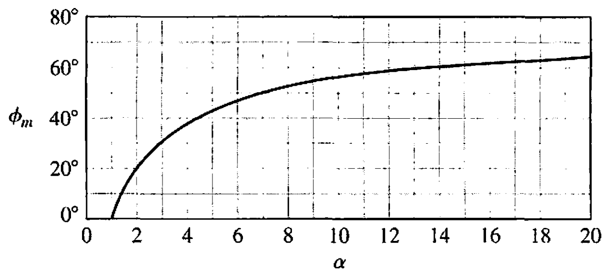


FIGURE 10.5 Maximum phase angle ϕ_m versus α for a phase-lead network.

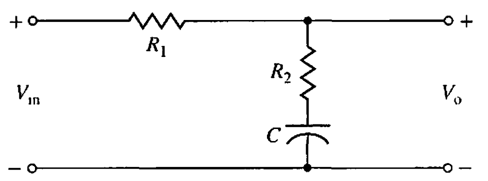


FIGURE 10.6 Phase-lag network.

It is often useful to add a cascade compensation network that provides a phase-lag characteristic. The **phase-lag network** is shown in Figure 10.6. The transfer function of the phase-lag network is

$$G_c(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{R_2 + 1/(Cs)}{R_1 + R_2 + 1/(Cs)} = \frac{R_2Cs + 1}{(R_1 + R_2)Cs + 1}. \tag{10.12}$$

When $\tau = R_2C$ and $\alpha = (R_1 + R_2)/R_2$, we have the **phase-lag compensation** transfer function

$$\boxed{G_c(s) = \frac{1 + \tau s}{1 + \alpha\tau s} = \frac{1}{\alpha}\frac{s + z}{s + p},} \tag{10.13}$$

where $z = 1/\tau$ and $p = 1/(\alpha\tau)$. In this case, because $\alpha > 1$, the pole lies closest to the origin of the s-plane, as shown in Figure 10.7. This type of compensation network is often called an integrating network because it has a frequency response like an integrator over a finite range of frequencies. The Bode diagram of the phase-lag network is obtained from the transfer function

$$G_c(j\omega) = \frac{1 + j\omega\tau}{1 + j\omega\alpha\tau} \tag{10.14}$$

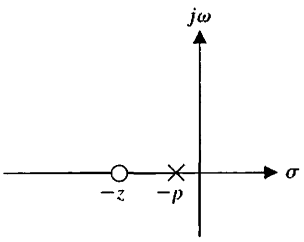


FIGURE 10.7 Pole–zero diagram of the phase-lag network.

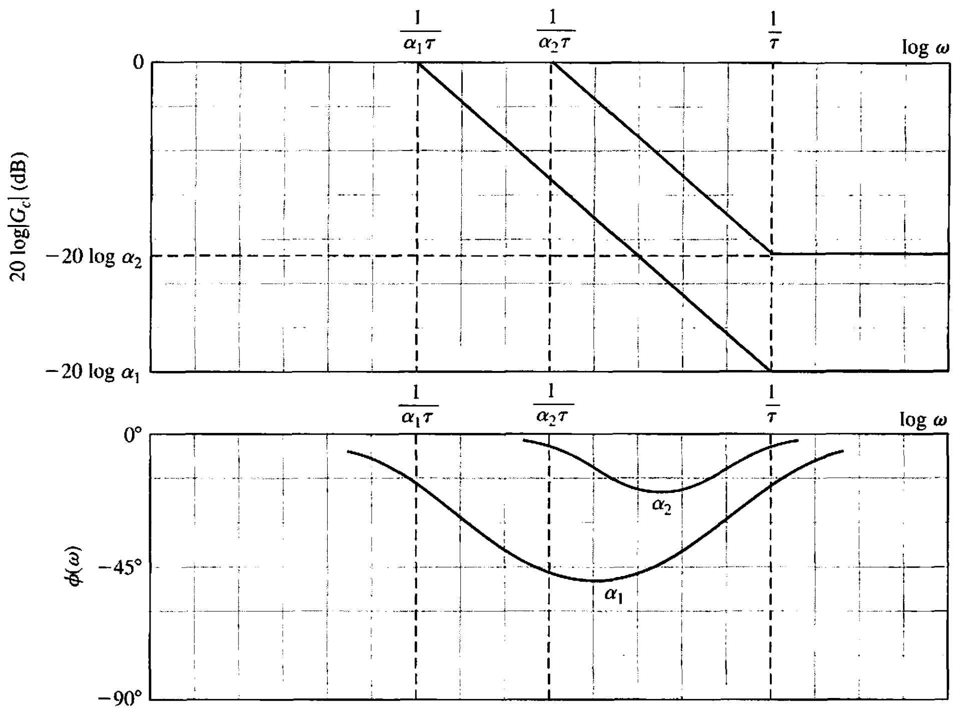


FIGURE 10.8 Bode diagram of the phase-lag network.

and is shown in Figure 10.8. The form of the Bode diagram of the **lag network** is similar to that of the phase-lead network; the difference is the resulting attenuation and phase-lag angle instead of amplification and phase-lead angle. However, note that the shapes of the diagrams of Figures 10.3 and 10.8 are similar. Therefore, we can show that the maximum phase lag occurs at $\omega_m = \sqrt{zp}$.

In the succeeding sections, we wish to utilize these compensation networks to obtain a desired system frequency response or s-plane root location. The **lead network** can provide a phase-lead angle and thus a satisfactory phase margin for a system. Alternatively, the phase-lead network can enable us to reshape the root locus and thus provide the desired root locations. The phase-lag network is used, not to provide a phase-lag angle, which is normally a destabilizing influence, but rather to provide an attenuation and to increase the steady-state error constant [3]. The following six sections discuss these approaches to design utilizing the phase-lead and phase-lag networks.

10.4 PHASE-LEAD DESIGN USING THE BODE DIAGRAM

The Bode diagram is used to design a suitable phase-lead network in preference to other frequency response plots. The frequency response of the cascade compensation network is added to the frequency response of the uncompensated system. That is, because the total loop transfer function of Figure 10.1(a) is $L(j\omega) = G_c(j\omega)G(j\omega)H(j\omega)$, we will first plot the Bode diagram for $G(j\omega)H(j\omega)$. Then we can examine the plot for

$G(j\omega)H(j\omega)$ and determine a suitable location for p and z of $G_c(j\omega)$ in order to satisfactorily reshape the frequency response. The uncompensated $G(j\omega)H(j\omega)$ is plotted with the desired gain to allow an acceptable steady-state error. Then the phase margin and the expected $M_{p\omega}$ are examined to find whether they satisfy the specifications. If the phase margin is not sufficient, phase lead can be added to the phase-angle curve of the system by placing the $G_c(j\omega)$ in a suitable location. To obtain maximum additional phase lead, we adjust the network so that the frequency ω_m is located at the frequency where the magnitude of the compensated magnitude curve crosses the 0-dB axis. (Recall the definition of phase margin.) The value of the added phase lead required allows us to determine the necessary value for α from Equation (10.11) or Figure 10.5. The zero $z = 1/(\alpha\tau)$ is located by noting that the maximum phase lead should occur at $\omega_m = \sqrt{zp}$, halfway between the pole and the zero. Because the total magnitude gain for the network is $20 \log \alpha$, we expect a gain of $10 \log \alpha$ at ω_m. Thus, we determine the compensation network by completing the following steps:

1. Evaluate the uncompensated system phase margin when the error constants are satisfied.
2. Allowing for a small amount of safety, determine the necessary additional phase lead ϕ_m.
3. Evaluate α from Equation (10.11).
4. Evaluate $10 \log \alpha$ and determine the frequency where the uncompensated magnitude curve is equal to $-10 \log \alpha$ dB. Because the compensation network provides a gain of $10 \log \alpha$ at ω_m, this frequency is the new 0-dB crossover frequency and ω_m simultaneously.
5. Calculate the pole $p = \omega_m\sqrt{\alpha}$ and $z = p/\alpha$.
6. Draw the compensated frequency response, check the resulting phase margin, and repeat the steps if necessary. Finally, for an acceptable design, raise the gain of the amplifier in order to account for the attenuation $(1/\alpha)$.

EXAMPLE 10.1 **A lead compensator for a type-two system**

Let us consider a single-loop feedback control system as shown in Figure 10.1(a), where

$$G(s) = \frac{K_1}{s^2} \tag{10.15}$$

and $H(s) = 1$. The uncompensated system is a type-two system and at first appears to possess a satisfactory steady-state error for both step and ramp input signals. However, the response of the uncompensated system is an undamped oscillation because

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_1}{s^2 + K_1}. \tag{10.16}$$

Therefore, the compensation network is added so that the loop transfer function is $L(s) = G_c(s)G(s)$. The specifications for the system are

$$\text{Settling time, } T_s \le 4 \text{ s;}$$
$$\text{System damping constant } \zeta \ge 0.45.$$

The settling time (with a 2% criterion) requirement is

$$T_s = \frac{4}{\zeta\omega_n} = 4;$$

therefore,

$$\omega_n = \frac{1}{\zeta} = \frac{1}{0.45} = 2.22.$$

Perhaps the easiest way to check the value of ω_n for the frequency response is to relate ω_n to the bandwidth ω_B, and evaluate the −3-dB bandwidth of the closed-loop system. For a closed-loop system with $\zeta = 0.45$, we estimate from Figure 8.26 that $\omega_B = 1.33\omega_n$. Therefore, we require a closed-loop bandwidth $\omega_B = 1.33(2.22) = 3.00$. The bandwidth can be checked following compensation by utilizing the Nichols chart. For the uncompensated system, the bandwidth of the system is $\omega_B = 1.33\omega_n$ and $\omega_n = \sqrt{K}$. Therefore, a loop gain equal to $K = \omega_n^2 \approx 5$ would be sufficient. To provide a suitable margin for the settling time, we will select $K = 10$ in order to draw the Bode diagram of

$$G(j\omega) = \frac{K}{(j\omega)^2}.$$

The Bode diagram of the uncompensated system is shown as solid lines in Figure 10.9.

By using Equation (9.58), the phase margin of the system is required to be approximately

$$\phi_{\text{pm}} = \frac{\zeta}{0.01} = \frac{0.45}{0.01} = 45°. \tag{10.17}$$

The phase margin of the uncompensated system is 0° because the double integration results in a constant 180° phase lag. Therefore, we must add a 45° phase-lead angle at the crossover (0-dB) frequency of the compensated magnitude curve. Evaluating the value of α, we have

$$\frac{\alpha - 1}{\alpha + 1} = \sin \phi_m = \sin 45°, \tag{10.18}$$

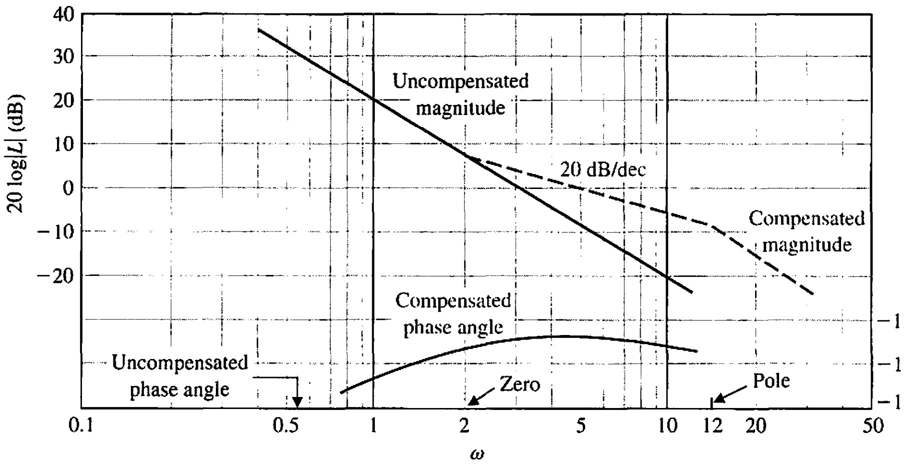


FIGURE 10.9
Bode diagram for Example 10.1.

and thus $\alpha = 5.8$. To provide a margin of safety, we will use $\alpha = 6$. The value of $10 \log \alpha$ is then equal to 7.78 dB. Then the lead network will add an additional gain of 7.78 dB at the frequency ω_m, and we want to have ω_m equal to the compensated slope near the 0-dB axis (the dashed line) so that the new crossover is ω_m and the dashed magnitude curve is 7.78 dB above the uncompensated curve at the crossover frequency. Thus, the compensated crossover frequency is located by evaluating the frequency where the uncompensated magnitude curve is equal to -7.78 dB, which in this case is $\omega = 4.95$. Then the maximum phase-lead angle is added to $\omega = \omega_m = 4.95$, as shown in Figure 10.9. Using step 5, we determine the pole $p = \omega_m\sqrt{\alpha} = 12.0$ and the zero $z = p/\alpha = 2.0$.

The bandwidth of the compensated system can be obtained from the Nichols chart. For estimating the bandwidth, we can simply examine Figure 9.26 and note that the -3-dB line for the closed-loop system occurs when the magnitude of $G(j\omega)$ is -6 dB and the phase shift of $G(j\omega)$ is approximately $-140°$. Therefore, to estimate the bandwidth from the open-loop diagram, we will approximate the bandwidth as the frequency for which $20 \times \log|G|$ is equal to -6 dB. Thus, the bandwidth of the uncompensated system is approximately equal to $\omega_B = 4.4$, while the bandwidth of the compensated system is equal to $\omega_B = 8.4$. The lead compensation doubles the bandwidth in this case, and satisfies the specification that $\omega_B > 3.00$. Therefore, the compensation of the system is completed, and the system specifications are satisfied. The total compensated loop transfer function is

$$L(j\omega) = G_c(j\omega)G(j\omega) = \frac{10[j\omega/2.0 + 1]}{(j\omega)^2[j\omega/12.0 + 1]}. \tag{10.19}$$

The transfer function of the compensator is

$$G_c(s) = \frac{1 + \alpha\tau s}{\alpha(1 + \tau s)} = \frac{1}{6}\frac{1 + s/2.0}{1 + s/12.0}, \tag{10.20}$$

in the form of Equation (10.8). Because an attenuation of $\frac{1}{6}$ results from the passive *RC* network, the gain of the amplifier in the loop must be raised by a factor of 6 so that the total DC loop gain is still equal to 10, as required in Equation (10.19). When we add the compensation network Bode diagram to the uncompensated Bode diagram, as in Figure 10.9, we assume that we can raise the amplifier gain to account for this $1/\alpha$ attenuation. The pole and zero values can be read from Figure 10.9, noting that $p = \alpha z$.

The total loop transfer function is (recall that $H(s) = 1$)

$$L(s) = \frac{10(1 + s/2)}{s^2(1 + s/12)} = \frac{60(s + 2)}{s^2(s + 12)}.$$

The closed-loop transfer function is

$$T(s) = \frac{60(s + 2)}{s^3 + 12s^2 + 60s + 120} \approx \frac{60(s + 2)}{(s^2 + 6s + 20)(s + 6)},$$

and the effects of the zero at $s = -2$ and the third pole at $s = -6$ will affect the transient response. Plotting the step response, we find an overshoot of 34% and a settling time of 1.4 seconds. ■

EXAMPLE 10.2 **A lead compensator for a second-order system**

A unity feedback control system has a loop transfer function

$$L(s) = \frac{K}{s(s + 2)}, \tag{10.21}$$

where $L(s) = G_c(s)G(s)$ and $H(s) = 1$. We want to have a steady-state error for a ramp input equal to 5% of the velocity of the ramp. Therefore, we require that

$$K_v = \frac{A}{e_{ss}} = \frac{A}{0.05A} = 20. \tag{10.22}$$

Furthermore, we desire that the phase margin of the system be at least 45°. The first step is to plot the Bode diagram of the uncompensated transfer function

$$G(j\omega) = \frac{K_v}{j\omega(0.5j\omega + 1)} = \frac{20}{j\omega(0.5j\omega + 1)}, \tag{10.23}$$

as shown in Figure 10.10(a). The frequency at which the magnitude curve crosses the 0-dB line is 6.2 rad/s, and the phase margin at this frequency is determined readily from the equation of the phase of $G(j\omega)$, which is

$$\underline{/G(j\omega)} = \phi(\omega) = -90° - \tan^{-1}(0.5\omega). \tag{10.24}$$

At the crossover frequency $\omega = \omega_c = 6.2$ rad/s, we have

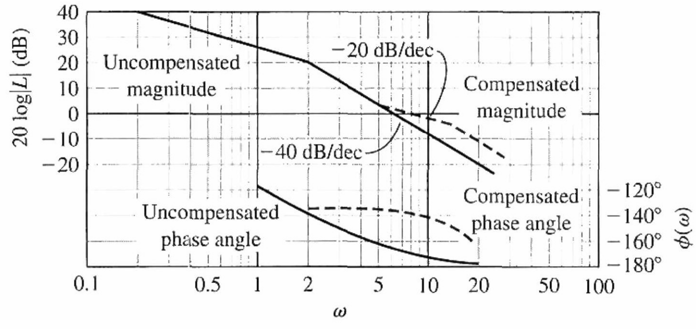
$$\phi(\omega) = -162°, \tag{10.25}$$

and therefore the phase margin is 18°. Using Equation (10.24) to evaluate the phase margin is often easier than drawing the complete phase-angle curve, which is shown in Figure 10.10(a). Thus, we need to add a phase-lead network so that the phase margin is raised to 45° at the new crossover (0-dB) frequency. Because the compensation crossover frequency is greater than the uncompensated crossover frequency, the phase lag of the uncompensated system is also greater. We shall account for this additional phase lag by attempting to obtain a maximum phase lead of $45° - 18° = 27°$, plus a small increment (10%) of phase lead to account for the added lag. Thus, we will design a compensation network with a maximum phase lead equal to $27° + 3° = 30°$. Then, calculating α, we obtain

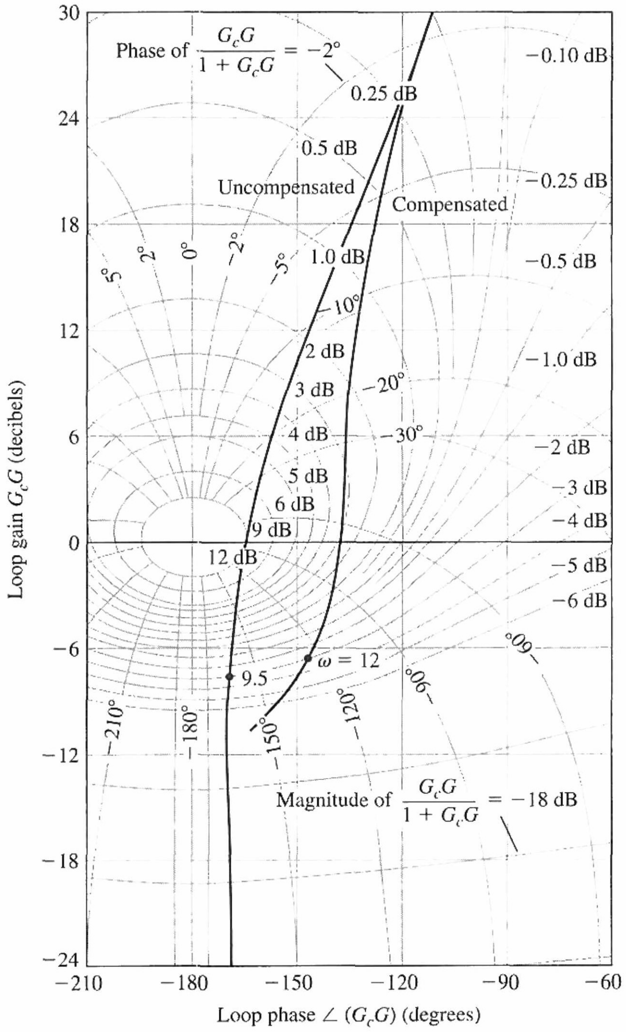
$$\frac{\alpha - 1}{\alpha + 1} = \sin 30° = 0.5, \tag{10.26}$$

and therefore $\alpha = 3$.

The maximum phase lead occurs at ω_m, and this frequency will be selected so that the new crossover frequency and ω_m coincide. The magnitude of the lead network at ω_m is $10 \log \alpha = 10 \log 3 = 4.8$ dB. The compensated crossover frequency is then evaluated where the magnitude of $G(j\omega)$ is -4.8 dB, and thus $\omega_m = \omega_c = 8.4$. Drawing the compensated magnitude line so that it intersects the



(a)



(b)

FIGURE 10.10
(a) Bode diagram for Example 10.2.
(b) Nichols diagram for Example 10.2.

0-dB axis at $\omega = \omega_c = 8.4$, we find that $z = \omega_m/\sqrt{\alpha} = 4.8$ and $p = \alpha z = 14.4$. Therefore, the compensation network is

$$G_c(s) = \frac{1}{3}\frac{1 + s/4.8}{1 + s/14.4}. \tag{10.27}$$

The total DC loop gain must be raised by a factor of three in order to account for the factor $1/\alpha = \frac{1}{3}$. Then the compensated loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{20(s/4.8 + 1)}{s(0.5s + 1)(s/14.4 + 1)}. \tag{10.28}$$

To verify the final phase margin, we can evaluate the phase of $G_c(j\omega)G(j\omega)$ at $\omega = \omega_c = 8.4$ and thus obtain the phase margin. The phase angle is then

$$\begin{aligned}\phi(\omega_c) &= -90^\circ - \tan^{-1} 0.5\omega_c - \tan^{-1}\frac{\omega_c}{14.4} + \tan^{-1}\frac{\omega_c}{4.8} \\ &= -90^\circ - 76.5^\circ - 30.0^\circ + 60.2^\circ \\ &= -136.3^\circ. \end{aligned} \tag{10.29}$$

Therefore, the phase margin for the compensated system is 43.7°. If we desire to have exactly a 45° phase margin, we would repeat the steps with an increased value of α—for example, with $\alpha = 3.5$. In this case, the phase lag increased by 7° between $\omega = 6.2$ and $\omega = 8.4$, and therefore the allowance of 3° in the calculation of α was not sufficient. The step response of this system yields a 28% overshoot with a settling time of 0.75 second.

The Nichols diagram for the compensated and uncompensated system is shown in Figure 10.10(b). The reshaping of the frequency response locus is clear on this diagram. Note the increased phase margin for the compensated system as well as the reduced magnitude of $M_{p\omega}$, the maximum magnitude of the closed-loop frequency response. In this case, $M_{p\omega}$ has been reduced from an uncompensated value of +12 dB to a compensated value of approximately +3.2 dB. Also, we note that the closed-loop 3-dB bandwidth of the compensated system is equal to 12 rad/s compared with 9.5 rad/s for the uncompensated system. ■

Looking again at Examples 10.1 and 10.2, we note that the system design is satisfactory when the asymptotic curve for the magnitude $20 \log|GG_c|$ crosses the 0-dB line with a slope of −20 dB/decade.

10.5 PHASE-LEAD DESIGN USING THE ROOT LOCUS

The design of the phase-lead compensation network can also be readily accomplished using the root locus. The phase-lead network has a transfer function

$$G_c(s) = \frac{s + 1/\alpha\tau}{s + 1/\tau} = \frac{s + z}{s + p}, \tag{10.30}$$

where α and τ are defined for the RC network in Equation (10.7). The locations of the zero and pole are selected so as to result in a satisfactory root locus for the compensated system. The specifications of the system are used to specify the desired location of the dominant roots of the system. The s-plane root locus method is as follows:

1. List the system specifications and translate them into a desired root location for the dominant roots.
2. Sketch the uncompensated root locus, and determine whether the desired root locations can be realized with an uncompensated system.
3. If a compensator is necessary, place the zero of the phase-lead network directly below the desired root location (or to the left of the first two real poles).
4. Determine the pole location so that the total angle at the desired root location is 180° and therefore is on the compensated root locus.
5. Evaluate the total system gain at the desired root location and then calculate the error constant.
6. Repeat the steps if the error constant is not satisfactory.

Therefore, we first locate our desired dominant root locations so that the dominant roots satisfy the specifications in terms of ζ and ω_n, as shown in Figure 10.11(a). The root locus of the uncompensated system is sketched as illustrated in Figure 10.11(b).

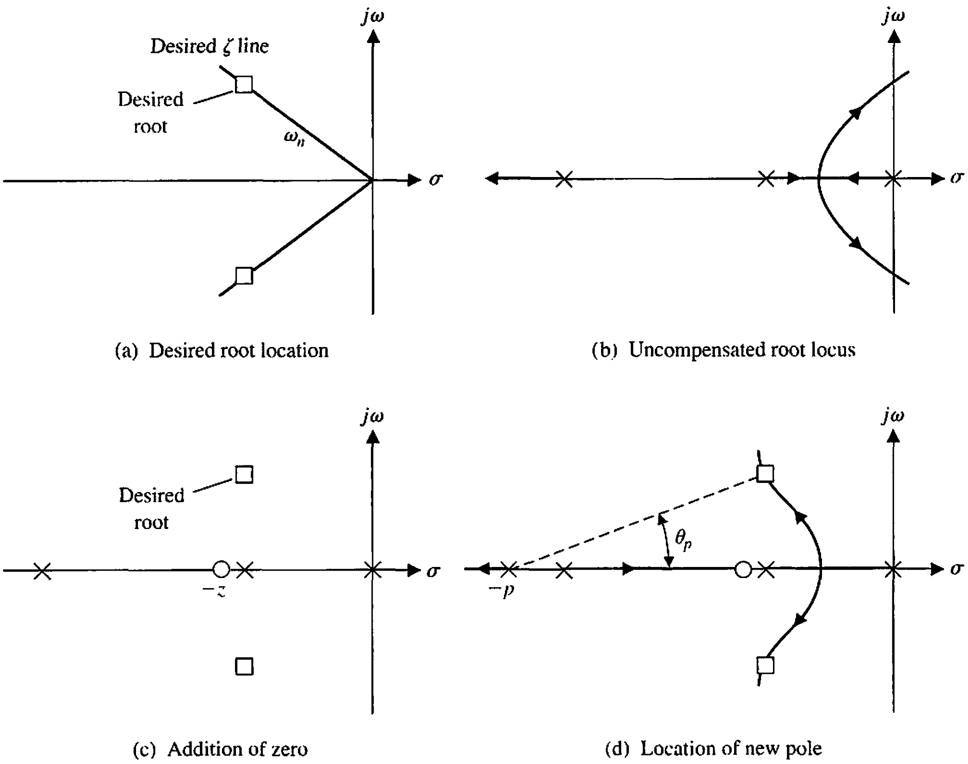


FIGURE 10.11 Compensation on the s-plane using a phase-lead network.

Then the zero is added to provide a phase lead by placing it to the left of the first two real poles. Some caution is necessary because the zero must not alter the dominance of the desired roots; that is, the zero should not be placed closer to the origin than the second pole on the real axis, or a real root near the origin will result and will dominate the system response. Thus, in Figure 10.11(c), we note that the desired root is directly above the second pole, and we place the zero z somewhat to the left of the second real pole.

Consequently, the real root may be near the real zero, and the coefficient of this term of the partial fraction expansion may be relatively small. Hence, the response due to this real root may have very little effect on the overall system response. Nevertheless, the designer must be continually aware that the compensated system response will be influenced by the roots and zeros of the system and that the dominant roots will not by themselves dictate the response. It is usually wise to allow for some margin of error in the design and to test the compensated system using a computer simulation.

Because the desired root is a point on the root locus when the final compensation is accomplished, we expect the algebraic sum of the vector angles to be 180° at that point. Thus, we calculate the angle θ_p from the pole of the compensator in order to result in a total angle of 180°. Then, locating a line at an angle θ_p intersecting the desired root, we are able to evaluate the compensator pole p, as shown in Figure 10.11(d).

The advantage of the root locus method is the ability of the designer to specify the location of the dominant roots and therefore the dominant transient response. The disadvantage of the method is that we cannot directly specify an error constant (for example, K_v) as in the Bode diagram approach. After the design is complete, we evaluate the gain of the system at the root location, which depends on p and z, and then calculate the error constant for the compensated system. If the error constant is not satisfactory, we must repeat the design steps and alter the location of the desired root as well as the location of the compensator pole and zero. We shall consider again Examples 10.1 and 10.2 and design a compensation network using the root locus (s-plane) approach.

EXAMPLE 10.3 **Lead compensator using the root locus**

Let us consider again the system of Example 10.1 where the uncompensated loop transfer function is

$$L(s) = \frac{K_1}{s^2}. \tag{10.31}$$

The characteristic equation of the uncompensated system is

$$1 + L(s) = 1 + \frac{K_1}{s^2} = 0, \tag{10.32}$$

and the root locus is the $j\omega$-axis. Therefore, we propose to compensate this system with a network

$$G_c(s) = \frac{s + z}{s + p}, \tag{10.33}$$

where $|z| < |p|$. The specifications for the system are

Settling time (with a 2% criterion), $T_s \leq 4$ s;

Percent overshoot for a step input $P.O. \leq 35\%$.

Therefore, the damping ratio should be $\zeta \geq 0.32$. The settling time requirement is

$$T_s = \frac{4}{\zeta\omega_n} = 4,$$

so $\zeta\omega_n = 1$. Thus, we will choose a desired dominant root location as

$$r_1, \hat{r}_1 = -1 \pm j2, \tag{10.34}$$

as shown in Figure 10.12 (hence, $\zeta = 0.45$).

Now we place the zero of the compensator directly below the desired location at $s = -z = -1$, as shown in Figure 10.12. Measuring the angle at the desired root, we have

$$\phi = -2(116^\circ) + 90^\circ = -142^\circ.$$

Therefore, to have a total of 180° at the desired root, we evaluate the angle from the undetermined pole, θ_p, as

$$-180^\circ = -142^\circ - \theta_p, \tag{10.35}$$

or $\theta_p = 38^\circ$. Then a line is drawn at an angle $\theta_p = 38^\circ$ intersecting the desired root location and the real axis, as shown in Figure 10.12. The point of intersection with the real axis is then $s = -p = -3.6$. Therefore, the compensator is

$$G_c(s) = \frac{s+1}{s+3.6}, \tag{10.36}$$

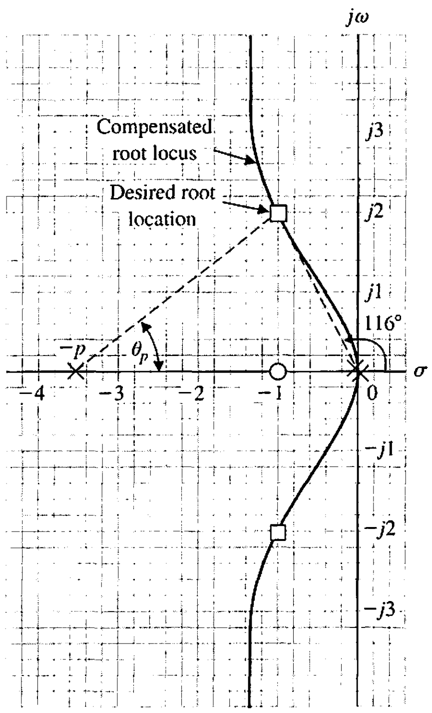


FIGURE 10.12 Phase-lead design for Example 10.3.

and the compensated loop transfer function for the system is

$$L(s) = G_c(s)G(s) = \frac{K_1(s + 1)}{s^2(s + 3.6)}. \tag{10.37}$$

The gain K_1 is evaluated by measuring the vector lengths from the poles and zeros to the root location. Hence,

$$K_1 = \frac{(2.23)^2(3.25)}{2} = 8.1. \tag{10.38}$$

Finally, the error constants of this system are evaluated. We find that this system with two open-loop integrations will result in a zero steady-state error for a step and ramp input signal. The acceleration constant is

$$K_a = \frac{8.1}{3.6} = 2.25. \tag{10.39}$$

The steady-state performance of this system is quite satisfactory, and therefore the compensation is complete. When we compare the compensation network evaluated by the s-plane method with the network obtained by using the Bode diagram approach, we find that the magnitudes of the poles and zeros are different. However, the resulting system will have the same performance, and we need not be concerned with the difference. In fact, the difference arises from the arbitrary design step (number 3), which places the zero directly below the desired root location. If we placed the zero at $s = -2.0$, we would find that the pole evaluated by the s-plane method is approximately equal to the pole evaluated by the Bode diagram approach.

The specifications for the transient response of this system were originally expressed in terms of the overshoot and the settling time of the system. These specifications were translated, on the basis of an approximation of the system by a second-order system, to an equivalent ζ and ω_n and therefore a desired root location. However, the original specifications will be satisfied only if the selected roots are dominant. The zero of the compensator and the root resulting from the addition of the compensator pole result in a third-order system with a zero. The validity of approximating this system with a second-order system without a zero is dependent upon the validity of the dominance assumption. Often, the designer will simulate the final design by using a digital computer and obtain the actual transient response of the system. In this case, a computer simulation of the system resulted in an overshoot of 46% and a settling time (to within 2% of the final value) of 3.8 seconds for a step input. These values compare moderately well with the specified values of 35% and 4 seconds, and they justify the use of the dominant root specifications. The difference in the overshoot from the specified value is due to the zero, which is not negligible. Thus, again we find that the specification of dominant roots is a useful approach but must be utilized with caution and understanding. A second attempt to obtain a compensated system with an overshoot of 30% would use a **prefilter** to eliminate the effect of the zero in the closed-loop transfer function, as described in Section 10.10. ■

EXAMPLE 10.4 **Lead compensator for a type-one system**

Now, let us consider again the system of Example 10.2 and design a compensator based on the root locus approach. The system loop transfer function is

$$L(s) = \frac{K}{s(s + 2)}. \tag{10.40}$$

We want the damping ratio of the dominant roots of the system to be $\zeta = 0.45$ and the velocity error constant to be equal to 20. To satisfy the error constant requirement, the gain of the uncompensated system must be $K = 40$. When $K = 40$, the roots of the uncompensated system are

$$s^2 + 2s + 40 = (s + 1 + j6.25)(s + 1 - j6.25). \tag{10.41}$$

The damping ratio of the uncompensated roots is approximately 0.16, and therefore a compensation network must be added. To achieve a rapid settling time, we will select the real part of the desired roots as $\zeta\omega_n = 4$, and therefore $T_s = 1$ s. This implies the natural frequency of these roots is fairly large, $\omega_n = 9$; hence, the velocity constant should be reasonably large. The location of the desired roots is shown in Figure 10.13(a) for $\zeta\omega_n = 4$, $\zeta = 0.45$, and $\omega_n = 9$.

The zero of the compensator is placed at $s = -z = -4$, directly below the desired root location. Then the angle at the desired root location is

$$\phi = -116^\circ - 104^\circ + 90^\circ = -130^\circ. \tag{10.42}$$

Therefore, the angle from the undetermined pole is determined from

$$-180^\circ = -130^\circ - \theta_p,$$

and thus $\theta_p = 50^\circ$. This angle is drawn to intersect the desired root location, and p is evaluated as $s = -p = -10.6$, as shown in Figure 10.13(a). The gain of the compensated system is then

$$K = \frac{9(8.25)(10.4)}{8} = 96.5. \tag{10.43}$$

The compensated system loop transfer function is then

$$L(s) = G_c(s)G(s) = \frac{96.5(s + 4)}{s(s + 2)(s + 10.6)}. \tag{10.44}$$

Therefore, the velocity constant of the compensated system is

$$K_v = \lim_{s\to 0} s[G_c(s)G(s)] = \frac{96.5(4)}{2(10.6)} = 18.2. \tag{10.45}$$

The velocity constant of the compensated system is less than the desired value of 20. Accordingly, we must repeat the design procedure for a second choice of a desired root. If we choose $\omega_n = 10$, the process can be repeated, and the resulting gain K will be increased. The compensator pole and zero location will also be altered.

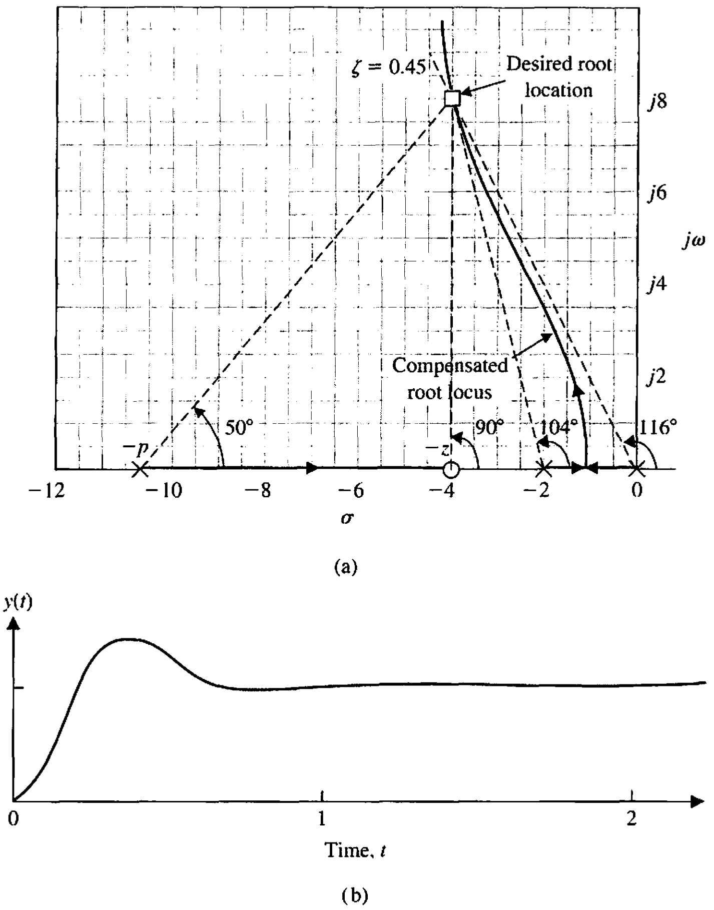


FIGURE 10.13 (a) Design of a phase-lead network on the s-plane for Example 10.4. (b) Step response of the compensated system of Example 10.4.

Then the velocity constant can be again evaluated. We will leave it as an exercise to show that for $\omega_n = 10$, the velocity constant is $K_v = 22.7$ when $z = 4.5$ and $p = 11.6$.

Finally, for the compensation network of Equation (10.44), we have

$$G_c(s) = \frac{s + 4}{s + 10.6} = \frac{s + 1/(\alpha\tau)}{s + 1/\tau}. \qquad (10.46)$$

The design of an RC-lead network to implement $G_c(s)$, as shown in Figure 10.4, follows directly from Equations (10.46) and (10.7):

$$G_c(s) = \frac{R_2}{R_1 + R_2} \frac{R_1Cs + 1}{[R_1R_2/(R_1 + R_2)]\,Cs + 1}. \qquad (10.47)$$

Thus, in this case, we have

$$\frac{1}{R_1C} = 4 \quad \text{and} \quad \alpha = \frac{R_1 + R_2}{R_2} = \frac{10.6}{4}.$$

Then, choosing $C = 1\ \mu f$, we obtain $R_1 = 250{,}000\ \Omega$ and $R_2 = 152{,}000\ \Omega$. The step response of the compensated system yields a 32% overshoot with a settling time of