3.3 linear Independent

ex consider the vector in $\mathbb{R}^3$.

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}.$$

independent ??

$$x_3 = 3x_1 + 2x_2$$

we write x_3 as (a linear combination of x_1, x_2)

$\{x_1, x_2, x_3\}$ dependent.

because of this linear combination

$$3x_1 = x_3 - 2x_2$$

$$x_1 = \frac{1}{3}x_3 - \frac{2}{3}x_2.$$

$x_2 =$ ______

if we cant write like this relation → Linearly independent.

$$\boxed{\begin{aligned} \text{span}(x_1, x_2, x_3) &= \text{span}(x_1, x_2) = \text{span}(x_1, x_3) \\ &= \text{span}(x_2, x_3) \end{aligned}}$$

ex $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}\right)$

($v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$)

$4v_1 = v_2$ → linearly dependent.

zero vector → dependent set

Def:

(1) The vectors $v_1, v_2, \ldots v_n$ in a vector space (V) are said to be **linearly independent** If

vector

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

then $c_1, c_2, c_3 \ldots$ must equal zero.

(2) $v_1, v_2, \ldots v_n$ are said to be **linearly dependent** If

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

then there exist scalars $c_1, \ldots, c_n$ not all zeros.

ex $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ linearly independent?

Sol:

$$c_1v_1 + c_2v_2 = 0$$

$$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$, find c_1, c_2.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$\rightarrow$ $c_2 = 0$

$c_1 + c_2 = 0 \rightarrow c_1 + 0 = 0 \rightarrow \boxed{c_1 = 0}$

$c_1 = c_2 = 0 \Rightarrow$ the vectors are linearly independent

ex $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. $c_1v_1 = 0 \Rightarrow c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \rightarrow c_1 = 0$

$\rightarrow$ ~~[illegible]~~ indep.

ex which of the following sets are linearly independent in R^3.

(a) $\{ (1,1,1)^T, (1,1,0)^T, (1,0,0)^T \}$

sol:
$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

find c_1, c_2, c_3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \to \boxed{c_1 = 0}$$

$c_1 + c_2 = 0 \to$

$0 + c_2 = 0 \to \boxed{c_2 = 0}$

$\underset{0}{c_1} + \underset{0}{c_2} + c_3 = 0 \to \boxed{c_3 = 0}$

$c_1, c_2, c_3 = 0 \to$ the vectors are linearly independent.

(b) $\{ (1,0,1)^T, (0,1,0)^T \} \to$ حلو L. indep.

(c) $\{ (1,2,4)^T, (2,1,3)^T, (4,-1,1)^T \}$

sol:
$$c_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ find } c_1, c_2, c_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 1 & -1 & 0 \\ 4 & 3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -3 & -9 & 0 \\ 0 & -5 & -15 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -5 & -15 & 0 \end{array}\right] \rightarrow$$

$$\begin{array}{c} \begin{array}{ccc} c_1\,\cancel{x_1} & c_2\,\cancel{x_2} & c_3\,\cancel{x_3} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

c_3 : free variable $\rightarrow$ $c_3 = \alpha$, $\alpha \in \mathbb{R}$

$\rightarrow$ infinitly may sol. $c_3 \neq 0$

$\rightarrow$ the vectors are linearly dependent

* write V_3 in terms of V_1, V_2 ?

Sol:

solve the system: $\boxed{c_3 = \alpha}$

$c_2 + 3c_3 = 0 \Rightarrow c_2 + 3\alpha = 0 \rightarrow \boxed{c_2 = -3\alpha}$

$c_1 + 2c_2 + 4c_3 = 0 \rightarrow c_1 + 2(-3\alpha) + 4(\alpha) = 0$

$c_1 - 6\alpha + 4\alpha = 0 \rightarrow \boxed{c_1 = 2\alpha}$

$\Rightarrow c_1V_1 + c_2V_2 + c_3V_3 = 0$ (let $\alpha = 1$)

$2V_1 + -3V_2 + V_3 = 0$

$\Rightarrow \boxed{V_3 = -2V_1 + 3V_2}$

$c_1v_1 + c_2v_2 + c_3v_3 \cdots = 0$

homo. $\longrightarrow$ unique sol. (zero sol.) (trivial sol.)

$c_1 = c_2 = \cdots = c_n = 0$

or

$\longrightarrow$ infinitly many sol. (trivial sol. + non-trivial)

(α, β)

infinitly many sol.

(حل وحيد فقط) unique.

Thm طريقة اخرى لتعين linearly independent مع شرط معين

$x_1, x_2, \ldots x_n$ are (n) vectors in $\boxed{R^n}$.

$X = (x_1, x_2, \ldots, x_n)$.

X is singular $\Longleftrightarrow$ $x_1, x_2, \ldots x_n$ are linearly dependent

$(\det = 0)$

X is nonsingular $\Leftrightarrow$ $x_1, x_2, \ldots x_n$ are linearly independent.

$(\det \neq 0)$

ex Determine whether the vectors $(4,2,3)^T$, $(2,3,1)^T$, $(2,-5,3)^T$ are linearly dependent ??

sol: $v_1, v_2, v_3 \in R^3$, $3 =$ عددهم

$$X = \begin{vmatrix} 4 & 2 & 2 \\ 2 & 3 & -5 \\ 3 & 1 & 3 \end{vmatrix}$$

$$= 4\begin{vmatrix} 3 & -5 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & -5 \\ 3 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 4(14) - 2(21) + 2(-7) = 56 - 42 - 14$$

$$= 56 - 56 = 0$$

$\det = 0 \longrightarrow$ the vectors are linearly dependent.

ex $x_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$, $x_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ -2 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 0 \\ 7 \\ 7 \end{bmatrix}$

check linearly independent ?

sol:

$$\begin{vmatrix} 1 & -2 & 1 \\ -1 & 3 & 0 \\ 2 & 1 & 7 \\ 3 & -2 & 7 \end{vmatrix}_{4\times 3}$$

نعود للطريقة الأولى $\longrightarrow$ $c_1x_1 + c_2x_2 + c_3x_3 = 0$

$\downarrow$

free variable $\rightarrow$ linearly dependent.

$R^4 \rightarrow 3 =$ العدد $\propto$

فشل الطريقة الثانية

P133 واجب

Thm

$v_1, v_2, \dots v_n$ are vectors in a vector space V.

A vector $v \in \text{span}(v_1, v_2, \dots v_n)$ can be written uniquely as a linear combination of $v_1, v_2, \dots v_n$ if and only if $v_1, v_2, \dots v_n$ are $\longleftrightarrow$ linearly independent.

ex In $\mathbb{R}^2$, $\{e_1, e_2\}$ check L. independent

$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \in V$ شو الثوابت $\begin{bmatrix} a \\ b \end{bmatrix}$.

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 3e_1 + 4e_2$$ طريقة واحدة فقط

$\rightarrow$ $\{e_1, e_2\}$ L. independent.

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vectors بطرق أكثر من ال عكسه

$$\left.\begin{aligned} \begin{bmatrix} 3 \\ 4 \end{bmatrix} &= \_\_\_\_\_\_ \\ \begin{bmatrix} 3 \\ 4 \end{bmatrix} &= \_\_\_\_\_\_ \\ &= \_\_\_\_\_\_ \end{aligned}\right\} \text{L. dep.}$$

ex in P_3.

$P_1 = x^2 - 2x + 3$

$P_2 = 2x^2 + x + 8$

$P_3 = x^2 + 8x + 7$

check linearly independent.

Sol: $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$v_1 \to P_1$, $v_2 \to P_2$, $v_3 \to P_3$

$$c_1(x^2 - 2x + 3) + c_2(2x^2 + x + 8) + c_3(x^2 + 8x + 7) = 0x^2 + 0x + 0$$

توزيع

$$(c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3) = 0$$

$c_1 + 2c_2 + c_3 = 0$

$-2c_1 + c_2 + 8c_3 = 0$

$3c_1 + 8c_2 + 7c_3 = 0$

find c_1, c_2, c_3

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -2 & 1 & 8 & 0 \\ 3 & 8 & 7 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 10 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \to \begin{array}{cccc} c_1 & c_2 & c_3 & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \end{array}$$

$c_3 \to$ free $\longrightarrow c_3 \neq 0 \to$ linearly dependent

طريقة أخرى.

$$\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{vmatrix} = 0 \Rightarrow \det = 0$$

$\to$ singular $\to$ infinitly $\to$ nontrival sol.

$\longrightarrow$ linearly dependent

Def:

Let $f_1, f_2, \dots f_n$ be functions in $C^{(n-1)}[a,b]$.

then

Wronskian of $f_1, f_2, \dots f_n$ =

$$W(f_1, f_2, \dots f_n) = \begin{vmatrix} f_1(x) & f_2 & \dots & f_n \\ f_1' & f_2' & & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix}$$

بين قوسين تعني مشتقة

Thm

let $f_1, f_2, \dots f_n \in C^{(n-1)}[a,b]$,

If there exist a point x_0 in $[a,b]$ such that $W[f_1, f_2, \dots f_n](x_0) \neq 0$

then

$f_1, f_2, \dots f_n$ are linearly independent.

converse not true. X

العكس.

ex $\{e^x, e^{-x}\}$ check linearly independent ???

sol:

$$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}_{2\times 2}$$

$$-e^x e^{-x} - e^x e^{-x} = -e^0 - e^0 = -1-1 = -2$$

$W(e^x, e^{-x}) = -2 \neq 0$

$\rightarrow \{e^x, e^{-x}\}$ linearly independent.

ex $\{1, x, x^2, x^3\}$?? L.indep??

sol:

$$W(1, x, x^2, x^3) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} =$$

$= (1)(1)(2)(6) = 12 \neq 0 \rightarrow$ linearly independent.

Q5 $\{x_1, \longrightarrow x_k\}$ linearly independent

add $\longrightarrow$?? $\begin{cases} \text{indep.} \\ \text{dep.} \end{cases}$

delete $\longrightarrow$ indep.

$\{x_1 \longrightarrow x_k\}$ linearly dependent

add $\longrightarrow$

delete $\longrightarrow$