3.2 Subspaces فضاء جزئي

Def:
If S is a nonempty subset of a vector space V, then S is said to be a subspace of V if:

① $x, y \in S \longrightarrow x + y \in S$

② $x \in S$, $\alpha \in R \longrightarrow \alpha x \in S$

closure properties.

ex $S = \{ (x_1, x_2)^T : x_2 = 2x_1 \}$

is S a subspace of R^2.

Sol: ⓐ $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S \neq \phi$ (not empty).

اداة تربيع $= (x_1, x_2)^T = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$

$X = \begin{bmatrix} a \\ 2a \end{bmatrix}$, $Y = \begin{bmatrix} b \\ 2b \end{bmatrix}$.

① $X, Y \in S \xrightarrow{??} X + Y \in S$

$X + Y = \begin{bmatrix} a + b \\ 2a + 2b \end{bmatrix} = \begin{bmatrix} a + b \\ 2(a+b) \end{bmatrix} \in S$ ✓

② $x \in S, \alpha \in R \xrightarrow{??} \alpha X \in S$

$$\alpha X = \begin{bmatrix}\alpha a \\ \alpha 2a\end{bmatrix} = \begin{bmatrix}\alpha a \\ 2(\alpha a)\end{bmatrix} \in S$$

$\Rightarrow S$ is a subspace of R^2

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ex let $S = \{(x_1, x_2, 2)^T, x_1, x_2 \in R\}$

is S a subspace R^3. ?

sol, ① $(1,3,2)^T \in S \neq \phi$

$$\begin{bmatrix}x_1 \\ x_2 \\ 2\end{bmatrix}$$

$X = \begin{bmatrix}a \\ b \\ 2\end{bmatrix}, \quad Y = \begin{bmatrix}c \\ d \\ 2\end{bmatrix}$

① $X, Y \in S \xrightarrow{??} X + Y \in S$

$$X + Y = \begin{bmatrix}a+c \\ b+d \\ 4\end{bmatrix} \notin S$$

$\rightarrow S$ is not a subspace of R^3.

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ex $S = \{(x_1, x_2, x_3)^T : x_1 = x_2\}$ واجب

subspace of R^3 ??

sol. $\begin{bmatrix}x_1 \\ x_1 \\ x_3\end{bmatrix}$ $\quad X = \begin{bmatrix}a \\ a \\ b\end{bmatrix}, \quad Y = \begin{bmatrix}c \\ c \\ d\end{bmatrix}$

ex let $S = \{A \in R^{2\times2} : a_{12} = -a_{21}\}$

is S a subspace of $R^{2\times2}$??

sol: ① $\begin{bmatrix} 3 & 5 \\ -5 & 4 \end{bmatrix} \in S \neq \phi$

$a_{12} = -a_{21} \implies A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$X = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}, Y = \begin{bmatrix} d & e \\ -e & f \end{bmatrix}$

① $X+Y \in S \xrightarrow{??} X+Y \in S$

$X+Y = \begin{bmatrix} a+d & e+b \\ -(b+e) & c+f \end{bmatrix} \in S$ ✓

② $X \in S, \alpha \in R \xrightarrow{??} \alpha X \in S$

$\alpha X = \alpha \begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{bmatrix} \in S$ ✓

$\rightarrow S$ is a subspace of $R^{2\times2}$.

note

If S, T are two subspace of V, then

① $S \cap T$ is a subspace of V.

② $S \cup T$ is not a subspace of V.

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Q1 ⓓ $\{(x_1, x_2)^T, |x_1| = |x_2|\}$

is it a subspace of R^2 ??

sol: $\begin{bmatrix} 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \notin S$

$\in S \qquad \in S$

example on which the first condition is not satisfied.

$\rightarrow S$ is not a subspace of R^2.

Q3

(a) the set of all 2×2 diagonal matrices are subspace of $R^{2\times 2}$.

Sol: $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \in S \neq \phi$

$X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $Y = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

$X+Y \in S$, $\alpha X \in S$

$\rightarrow$ S is a subspace of $R^{2\times 2}$.

واجب Q1, Q3 c 3 P/125

Q All $n\times n$ symmetric matrices are subspace of $R^{n\times n}$.

Sol: --- $\neq \phi$.

$X, Y \in S$, X, Y symmetric. math

$X = X^T$, $Y = Y^T$.

(1) $X, Y \in S \xrightarrow{??} X+Y \in S$

$(X+Y)^T = X^T + Y^T = X + Y$

② $X \in S, \alpha \in R \xrightarrow{??} \alpha X \in S$

$(\alpha X)^T = \alpha X^T = \alpha X \in S$ ✓

$\rightarrow S$ is a subspace of $R^{n \times n}$

Null space of a matrix:

let A $m \times n$ matrix.

Null space of A $[N(A)]$ is the set of all solutions of the homogenous system $\underset{m \times n}{A} \underset{n \times 1}{X} = 0$

note $N(A)$ is a subspace of $R^{n \times 1} = R^n$

ex

Determine $N(A)$ If

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$$

sol:

the sol. of $AX = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \quad\; x_1 \quad\; x_2 \quad\; x_3 \quad\; x_4 \qquad \\ \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right] \end{array}$$

free variable :- x_3, x_4

$$\boxed{\begin{matrix} x_3 = \alpha \\ x_4 = \beta \end{matrix}}, \quad \alpha, \beta \in R.$$

$$x_2 + 2x_3 - x_4 = 0$$

$$x_2 + 2\alpha - \beta = 0 \Rightarrow \boxed{x_2 = -2\alpha + \beta}$$

$$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 + -2\alpha + \beta + \alpha = 0$$

$$\boxed{x_1 = \alpha - \beta}$$

$$N(A) = \left\{ \begin{bmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{bmatrix}, \alpha, \beta \in R \right.$$

$$N(A) = \left\{ \begin{bmatrix} \alpha \\ -2\alpha \\ \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} -\beta \\ \beta \\ 0 \\ \beta \end{bmatrix}, \alpha, \beta \in R \right\}$$

$$N(A) = \left\{ \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \alpha, \beta \in R \right\}$$

$$= \mathrm{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$