Lecture $\boxed{8}$ Chp $\boxed{3}$ Interpolation and polynomial approximation

Section $\boxed{3.1}$ Interpolation and Lagrange polynomial

السبب الرياضي لكل التقريبات من اقتران إلى كثير حدود لوجود النظرية التالية:

Thm: (Weirestrass approximation Thm)

Let f be defined and continuous on $[a,b]$ $\forall x \in [a,b]$

$\Rightarrow \exists$ a polynomial $P(x)$ s.t $|P(x) - f(x)| < \epsilon$ $\forall \epsilon > 0$.

Lagrange: (Thm) Let $x_0, x_1, \dots, x_n$ are $(n+1)$ distinct numbers and let f be a function whose values at these numbers are given $\Rightarrow \exists!$ (there exist a unique) polynomial $P(x)$ s.t. $P(x_k) = f(x_k)$, $k = 0, 1, \dots, n$. and $P(x)$ take the following form: $P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + \dots + L_n(x) f(x_n)$.

$\Rightarrow P(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$: This is called the Lagrange polynomial or the n^{th} interpolation polynomial.

where
$$L_k(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$$

$$= \prod_{\substack{j=0 \\ j\neq k}}^{n} (x-x_j) \Big/ \prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j)$$

: $\prod$: Product

مضروب

Ex: Find the 2^{nd} interpolation polynomial for $f(x) = \frac{1}{x}$

$x_0 = 2$, $x_1 = 2.5$ and $x_2 = 4$.

Sl: $P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$.

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} f(2) + \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} f(2.5) + \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)} f(4)$$

بعد التبسيط ينتج أن

$$P(x) = \left(\frac{1}{2} + \frac{-8}{15} + \frac{1}{2}\right)x^2 + \left(\frac{-6.5}{2} + \frac{48}{15} + \frac{-4.5}{12}\right)x + \left(? + \frac{-48}{15} + \frac{5}{12}\right)$$

Thm: suppose that $x_0, x_1, \dots, x_n$ are distinct numbers in $[a,b]$ and $f \in C[a,b] \Rightarrow \forall x \in [a,b]$ $\exists c \in (a,b)$ s.t.

$f(x) = P(x) + R(x)$. where

$$P(x) = \sum_{k=0}^{n} L_k(x) f(x_k) \quad \text{and}$$

$$R(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)(x-x_1)\dots(x-x_n)$$

$P(x)$: كثير الحدود الذي نقرب به $f(x)$

$R(x)$: الايرور الذي ينتج من التقريب

Ex: Let $f(x) = e^{x}$, $x_0 = 1$, $x_1 = 2$, $x_2 = 2.5$, $x_3 = 4$ and $x_4 = 5$.

Find $P_{2,3,4}(x)$.

3

Sol: $P_{2,3,4}(x) = L_2(x) f(x_2) + L_3(x) f(x_3) + L_4(x) f(x_4)$.

$$= \frac{(x - x_3)(x - x_4)}{(x_2 - x_3)(x_2 - x_4)} e^{x_2} + \frac{(x - x_2)(x - x_4)}{(x_3 - x_2)(x_3 - x_4)} e^{x_3} + \frac{(x - x_2)(x - x_3)}{(x_4 - x_2)(x_4 - x_3)} e^{x_4}$$

$$= \frac{(x-4)(x-5)}{(2.5-4)(2.5-5)} e^{2.5} + \frac{(x-2.5)(x-5)}{(4-2.5)(4-5)} e^{4} + \frac{(x-2.5)(x-4)}{(5-2.5)(5-4)} e^{5}$$

So, $$P_{2,3,4}(x) = \frac{(x-4)(x-5)}{(-1.5)(-2.5)} e^{2.5} + \frac{(x-2.5)(x-5)}{-1.5} e^{4} + \frac{(x-2.5)(x-4)}{2.5} e^{5}$$

Now, coefficient of $x^2 = \frac{e^{2.5}}{3.75} + \frac{e^{4}}{-1.5} + \frac{e^{5}}{2.5}$

coefficient of x: $\frac{-9 e^{2.5}}{3.75} + \frac{-7.5 e^{4}}{-1.5} + \frac{-6.5 e^{5}}{2.5}$

coefficient of x^0: $\frac{20 e^{2.5}}{3.75} + \frac{12.5 e^{4}}{-1.5} + \frac{10 e^{5}}{2.5}$

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