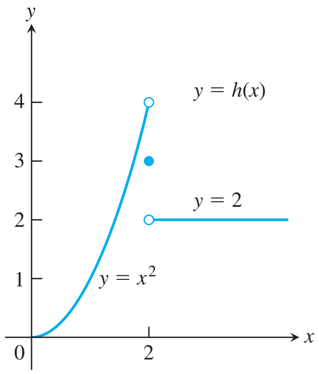
58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$



Show that

a. $\lim_{x\to 2} h(x) \neq 4$

b. $\lim_{x\to 2} h(x) \neq 3$

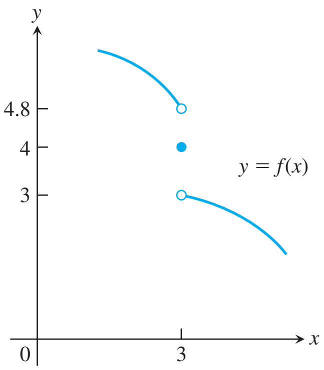
c. $\lim_{x\to 2} h(x) \neq 2$

59. For the function graphed here, explain why

a. $\lim_{x\to 3} f(x) \neq 4$

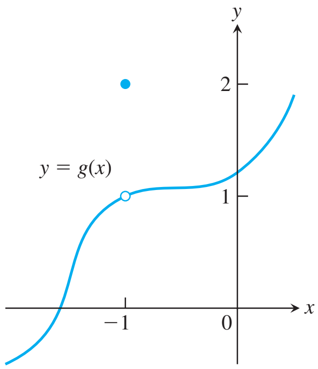
b. $\lim_{x\to 3} f(x) \neq 4.8$

c. $\lim_{x\to 3} f(x) \neq 3$



60. a. For the function graphed here, show that $\lim_{x\to -1} g(x) \neq 2$.

b. Does $\lim_{x\to -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



COMPUTER EXPLORATIONS

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

a. Plot the function $y = f(x)$ near the point c being approached.

b. Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.

c. Using the value $\epsilon = 0.2$, graph the banding lines $y_1 = L - \epsilon$ and $y_2 = L + \epsilon$ together with the function f near c.

d. From your graph in part (c), estimate a $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Test your estimate by plotting f, y_1, and y_2 over the interval $0 < |x - c| < \delta$. For your viewing window use $c - 2\delta \leq x \leq c + 2\delta$ and $L - 2\epsilon \leq y \leq L + 2\epsilon$. If any function values lie outside the interval $[L - \epsilon, L + \epsilon]$, your choice of δ was too large. Try again with a smaller estimate.

e. Repeat parts (c) and (d) successively for $\epsilon = 0.1, 0.05$, and 0.001.

61. $f(x) = \dfrac{x^4 - 81}{x - 3}, \quad c = 3$

62. $f(x) = \dfrac{5x^3 + 9x^2}{2x^5 + 3x^2}, \quad c = 0$

63. $f(x) = \dfrac{\sin 2x}{3x}, \quad c = 0$

64. $f(x) = \dfrac{x(1 - \cos x)}{x - \sin x}, \quad c = 0$

65. $f(x) = \dfrac{\sqrt[3]{x} - 1}{x - 1}, \quad c = 1$

66. $f(x) = \dfrac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}, \quad c = 1$

2.4 One-Sided Limits

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only.

Approaching a Limit from One Side

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. That is, f must be defined in some open interval about c, but not necessarily at c. Because of this, ordinary limits are called **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

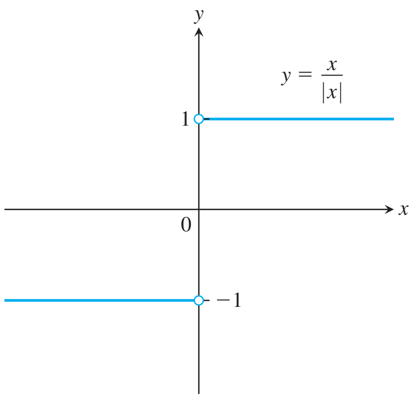


FIGURE 2.24 Different right-hand and left-hand limits at the origin.

The function $f(x) = x/|x|$ (Figure 2.24) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if $f(x)$ is defined on an interval (c, b), where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write

$$\lim_{x \to c^+} f(x) = L.$$

The symbol "$x \to c^+$" means that we consider only values of x greater than c.

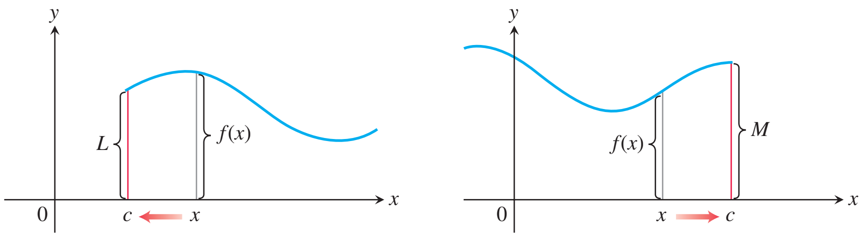
Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$

The symbol "$x \to c^-$" means that we consider only x-values less than c.

These informal definitions of one-sided limits are illustrated in Figure 2.25. For the function $f(x) = x/|x|$ in Figure 2.24 we have

$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$



(a) $\lim_{x \to c^+} f(x) = L$ (b) $\lim_{x \to c^-} f(x) = M$

FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

EXAMPLE 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.26. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have a two-sided limit at either -2 or 2 because each point does not belong to an open interval over which f is defined. ■

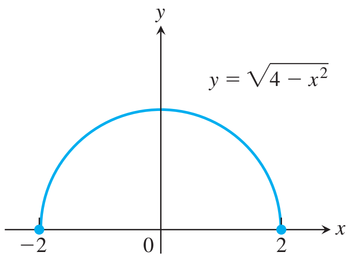


FIGURE 2.26 The function $f(x) = \sqrt{4 - x^2}$ has right-hand limit 0 at $x = -2$ and left-hand limit 0 at $x = 2$ (Example 1).

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as do the Sandwich Theorem and Theorem 5. One-sided limits are related to limits in the following way.

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

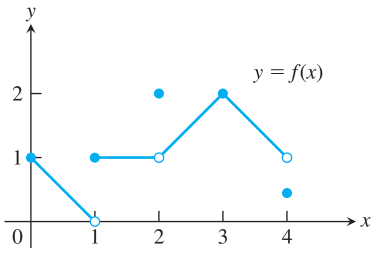


FIGURE 2.27 Graph of the function in Example 2.

EXAMPLE 2 For the function graphed in Figure 2.27,

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$,
$\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,
$\lim_{x \to 1^+} f(x) = 1$,
$\lim_{x \to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,
$\lim_{x \to 2^+} f(x) = 1$,
$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$,
$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

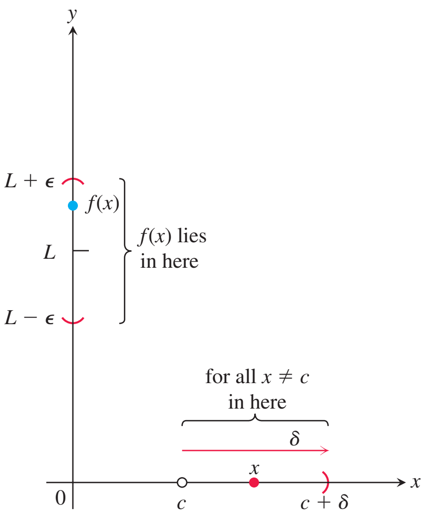


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

DEFINITIONS We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x \to c^+} f(x) = L \qquad \text{(see Figure 2.28)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c < x < c + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at c**, and write

$$\lim_{x \to c^-} f(x) = L \qquad \text{(see Figure 2.29)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c - \delta < x < c \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

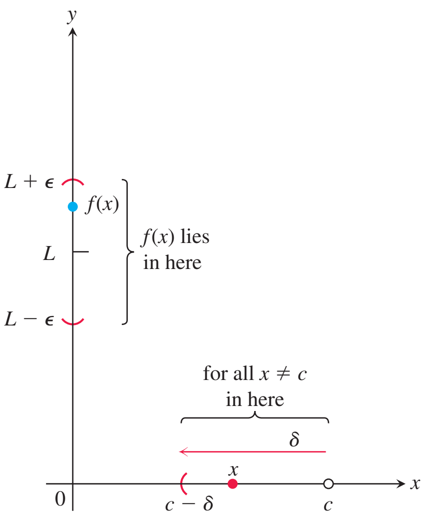


FIGURE 2.29 Intervals associated with the definition of left-hand limit.

EXAMPLE 3 Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution Let $\epsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x

$$0 < x < \delta \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon,$$

or

$$0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \epsilon.$$

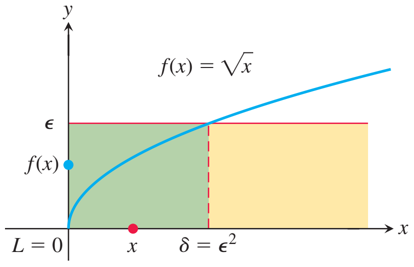


FIGURE 2.30 $\lim_{x \to 0^+} \sqrt{x} = 0$ in Example 3.

Squaring both sides of this last inequality gives

$$x < \epsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

If we choose $\delta = \epsilon^2$ we have

$$0 < x < \delta = \epsilon^2 \quad \Rightarrow \quad \sqrt{x} < \epsilon,$$

or

$$0 < x < \epsilon^2 \quad \Rightarrow \quad \left|\sqrt{x} - 0\right| < \epsilon.$$

According to the definition, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$ (Figure 2.30). ■

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case.

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.31).

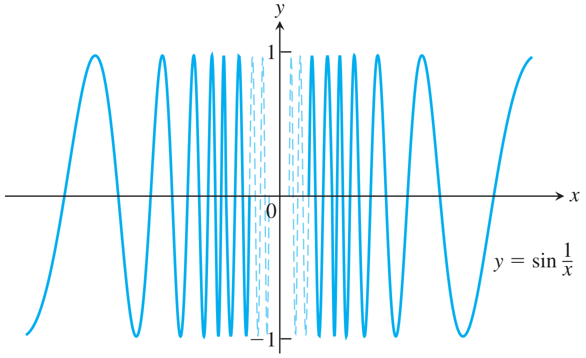


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y-axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

Limits Involving $(\sin \theta)/\theta$

A central fact about $(\sin \theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.32 and confirm it algebraically using the Sandwich Theorem. You will see the importance of this limit in Section 3.5, where instantaneous rates of change of the trigonometric functions are studied.

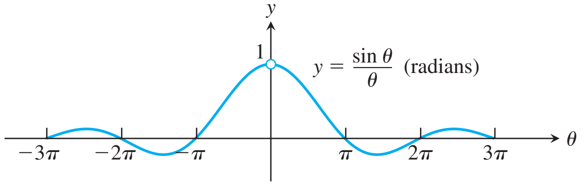


FIGURE 2.32 The graph of $f(\theta) = (\sin \theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

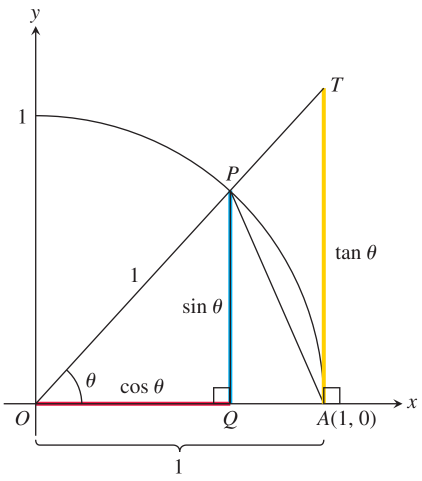


FIGURE 2.33 The figure for the proof of Theorem 7. By definition, $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

THEOREM 7—Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.33). Notice that

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \tag{2}$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

Equation (2) is where radian measure comes in: The area of sector OAP is $\theta/2$ only if θ is measured in radians.

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive, since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$ (Example 11b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin\theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.32). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ by Theorem 6. ∎

EXAMPLE 5 Show that **(a)** $\displaystyle\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ and **(b)** $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{5x} = \frac{2}{5}.$

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h}\\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta && \text{Let } \theta = h/2.\\ &= -(1)(0) = 0. && \text{Eq. (1) and Example 11a in Section 2.2}\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot\sin 2x}{(2/5)\cdot 5x}\\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} && \text{Now, Eq. (1) applies with } \theta = 2x.\\ &= \frac{2}{5}(1) = \frac{2}{5}\end{aligned}$$

EXAMPLE 6 Find $\lim_{t\to 0}\dfrac{\tan t\sec 2t}{3t}$.

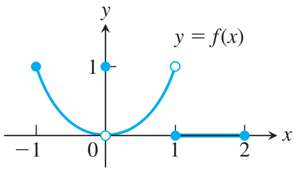
Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}\lim_{t\to 0}\frac{\tan t\sec 2t}{3t} &= \lim_{t\to 0}\frac{1}{3}\cdot\frac{1}{t}\cdot\frac{\sin t}{\cos t}\cdot\frac{1}{\cos 2t}\\ &= \frac{1}{3}\lim_{t\to 0}\frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t}\\ &= \frac{1}{3}(1)(1)(1) = \frac{1}{3}. && \text{Eq. (1) and Example 11b in Section 2.2}\end{aligned}$$

Exercises 2.4

Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to -1^+} f(x) = 1$ **b.** $\lim_{x\to 0^-} f(x) = 0$

c. $\lim_{x\to 0^-} f(x) = 1$ **d.** $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

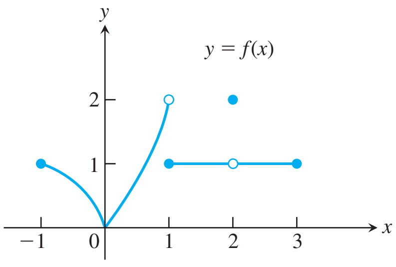
e. $\lim_{x\to 0} f(x)$ exists. **f.** $\lim_{x\to 0} f(x) = 0$

g. $\lim_{x\to 0} f(x) = 1$ **h.** $\lim_{x\to 1} f(x) = 1$

i. $\lim_{x\to 1} f(x) = 0$ **j.** $\lim_{x\to 2^-} f(x) = 2$

k. $\lim_{x\to -1^-} f(x)$ does not exist. **l.** $\lim_{x\to 2^+} f(x) = 0$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to -1^+} f(x) = 1$ **b.** $\lim_{x\to 2} f(x)$ does not exist.

c. $\lim_{x\to 2} f(x) = 2$ **d.** $\lim_{x\to 1^-} f(x) = 2$

e. $\lim_{x\to 1^+} f(x) = 1$ **f.** $\lim_{x\to 1} f(x)$ does not exist.

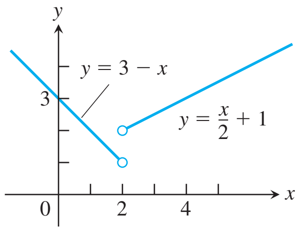
g. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

h. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$.

j. $\lim_{x\to -1^-} f(x) = 0$ **k.** $\lim_{x\to 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2. \end{cases}$



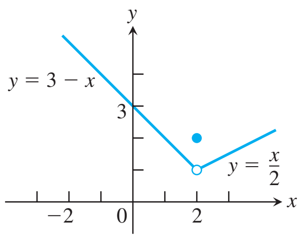
a. Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

d. Does $\lim_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



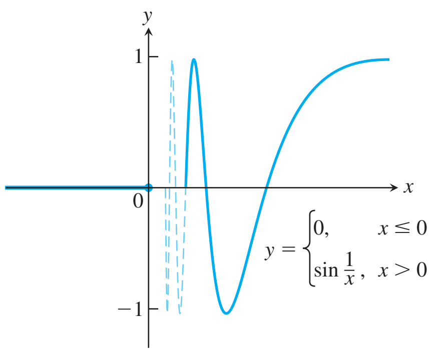
a. Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

d. Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\dfrac{1}{x}, & x > 0. \end{cases}$

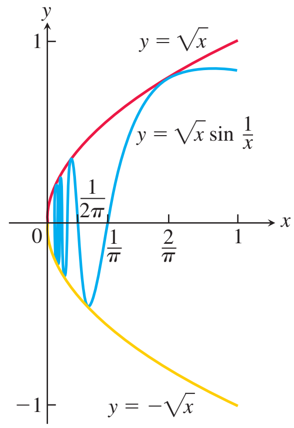


a. Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x}\sin(1/x)$.



a. Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

7. a. Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \ne 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x\to c} f(x)$ exist?

c. At what points does only the left-hand limit exist?

d. At what points does only the right-hand limit exist?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–18.

11. $\displaystyle\lim_{x\to -0.5^-} \sqrt{\frac{x + 2}{x + 1}}$

12. $\displaystyle\lim_{x\to 1^+} \sqrt{\frac{x - 1}{x + 2}}$

13. $\displaystyle\lim_{x\to -2^+} \left(\frac{x}{x + 1}\right)\left(\frac{2x + 5}{x^2 + x}\right)$

14. $\displaystyle\lim_{x\to 1^-} \left(\frac{1}{x + 1}\right)\left(\frac{x + 6}{x}\right)\left(\frac{3 - x}{7}\right)$

15. $\displaystyle\lim_{h\to 0^+} \frac{\sqrt{h^2 + 4h + 5} - \sqrt{5}}{h}$

16. $\lim_{h\to0^-}\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}$

17. a. $\lim_{x\to-2^+}(x+3)\frac{|x+2|}{x+2}$ b. $\lim_{x\to-2^-}(x+3)\frac{|x+2|}{x+2}$

18. a. $\lim_{x\to1^+}\frac{\sqrt{2x}(x-1)}{|x-1|}$ b. $\lim_{x\to1^-}\frac{\sqrt{2x}(x-1)}{|x-1|}$

Use the graph of the greatest integer function $y=\lfloor x\rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 19 and 20.

19. a. $\lim_{\theta\to3^+}\frac{\lfloor\theta\rfloor}{\theta}$ b. $\lim_{\theta\to3^-}\frac{\lfloor\theta\rfloor}{\theta}$

20. a. $\lim_{t\to4^+}(t-\lfloor t\rfloor)$ b. $\lim_{t\to4^-}(t-\lfloor t\rfloor)$

Using $\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1$

Find the limits in Exercises 21–42.

21. $\lim_{\theta\to0}\frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

22. $\lim_{t\to0}\frac{\sin kt}{t}$ (k constant)

23. $\lim_{y\to0}\frac{\sin 3y}{4y}$

24. $\lim_{h\to0^-}\frac{h}{\sin 3h}$

25. $\lim_{x\to0}\frac{\tan 2x}{x}$

26. $\lim_{t\to0}\frac{2t}{\tan t}$

27. $\lim_{x\to0}\frac{x\csc 2x}{\cos 5x}$

28. $\lim_{x\to0}6x^2(\cot x)(\csc 2x)$

29. $\lim_{x\to0}\frac{x+x\cos x}{\sin x\cos x}$

30. $\lim_{x\to0}\frac{x^2-x+\sin x}{2x}$

31. $\lim_{\theta\to0}\frac{1-\cos\theta}{\sin 2\theta}$

32. $\lim_{x\to0}\frac{x-x\cos x}{\sin^2 3x}$

33. $\lim_{t\to0}\frac{\sin(1-\cos t)}{1-\cos t}$

34. $\lim_{h\to0}\frac{\sin(\sin h)}{\sin h}$

35. $\lim_{\theta\to0}\frac{\sin\theta}{\sin 2\theta}$

36. $\lim_{x\to0}\frac{\sin 5x}{\sin 4x}$

37. $\lim_{\theta\to0}\theta\cos\theta$

38. $\lim_{\theta\to0}\sin\theta\cot 2\theta$

39. $\lim_{x\to0}\frac{\tan 3x}{\sin 8x}$

40. $\lim_{y\to0}\frac{\sin 3y\cot 5y}{y\cot 4y}$

41. $\lim_{\theta\to0}\frac{\tan\theta}{\theta^2\cot 3\theta}$

42. $\lim_{\theta\to0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

Theory and Examples

43. Once you know $\lim_{x\to a^+}f(x)$ and $\lim_{x\to a^-}f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a}f(x)$? Give reasons for your answer.

44. If you know that $\lim_{x\to c}f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+}f(x)$? Give reasons for your answer.

45. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to0^+}f(x)=3$ tell you anything about $\lim_{x\to0^-}f(x)$? Give reasons for your answer.

46. Suppose that f is an even function of x. Does knowing that $\lim_{x\to2^-}f(x)=7$ tell you anything about either $\lim_{x\to-2^-}f(x)$ or $\lim_{x\to-2^+}f(x)$? Give reasons for your answer.

Formal Definitions of One-Sided Limits

47. Given $\epsilon>0$, find an interval $I=(5,5+\delta)$, $\delta>0$, such that if x lies in I, then $\sqrt{x-5}<\epsilon$. What limit is being verified and what is its value?

48. Given $\epsilon>0$, find an interval $I=(4-\delta,4)$, $\delta>0$, such that if x lies in I, then $\sqrt{4-x}<\epsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 49 and 50.

49. $\lim_{x\to0^-}\frac{x}{|x|}=-1$

50. $\lim_{x\to2^+}\frac{x-2}{|x-2|}=1$

51. **Greatest integer function** Find **(a)** $\lim_{x\to400^+}\lfloor x\rfloor$ and **(b)** $\lim_{x\to400^-}\lfloor x\rfloor$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to400}\lfloor x\rfloor$? Give reasons for your answer.

52. **One-sided limits** Let $f(x)=\begin{cases}x^2\sin(1/x), & x<0\\ \sqrt{x}, & x>0.\end{cases}$

Find **(a)** $\lim_{x\to0^+}f(x)$ and **(b)** $\lim_{x\to0^-}f(x)$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to0}f(x)$? Give reasons for your answer.

2.5 Continuity

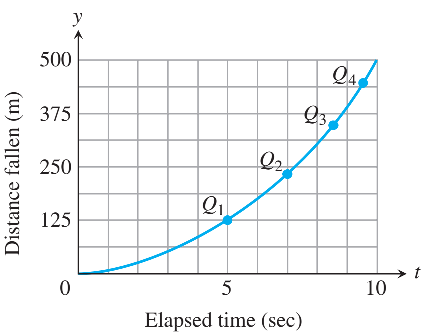


FIGURE 2.34 Connecting plotted points by an unbroken curve from experimental data $Q_1, Q_2, Q_3, \ldots$ for a falling object.

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure (Figure 2.34). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between. Intuitively, any function $y=f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.

Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.35, whose limits we investigated in Example 2 in the last section.