61. A function discontinuous at every point

a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

b. Is f right-continuous or left-continuous at any point?

62. If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

63. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

64. Discontinuous composite of continuous functions Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 9? Give reasons for your answer.

65. Never-zero continuous functions Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

66. Stretching a rubber band Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

67. A fixed point theorem Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

68. The sign-preserving property of continuous functions Let f be defined on an interval (a, b) and suppose that $f(c) \ne 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$.

69. Prove that f is continuous at c if and only if

$$\lim_{h \to 0} f(c + h) = f(c).$$

70. Use Exercise 69 together with the identities

$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$

$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solving Equations Graphically

T Use the Intermediate Value Theorem in Exercises 71–78 to prove that each equation has a solution. Then use a graphing calculator or computer grapher to solve the equations.

71. $x^3 - 3x - 1 = 0$

72. $2x^3 - 2x^2 - 2x + 1 = 0$

73. $x(x - 1)^2 = 1$ (one root)

74. $x^x = 2$

75. $\sqrt{x} + \sqrt{1 + x} = 4$

76. $x^3 - 15x + 1 = 0$ (three roots)

77. $\cos x = x$ (one root). Make sure you are using radian mode.

78. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

2.6 Limits Involving Infinity; Asymptotes of Graphs

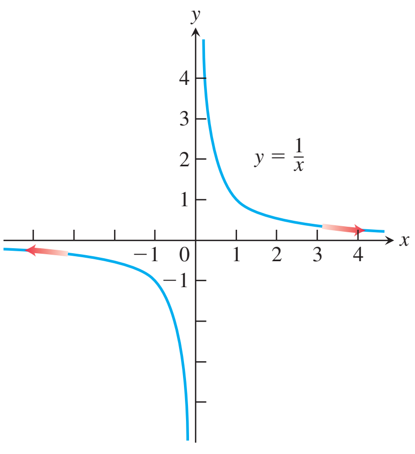

FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \to \pm\infty$. We further extend the concept of limit to *infinite limits*, which are not limits as before, but rather a new use of the term limit. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these limit ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \ne 0$ (Figure 2.49). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \to \infty$ or $x \to -\infty$, or that 0 is a *limit of $f(x) = 1/x$ at infinity and negative infinity*. Here are precise definitions.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

Intuitively, $\lim_{x\to\infty} f(x) = L$ if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L. Similarly, $\lim_{x\to-\infty} f(x) = L$ if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

The strategy for calculating limits of functions as $x \to \pm\infty$ is similar to the one for finite limits in Section 2.2. There we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying Theorem 1 on limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

The basic facts to be verified by applying the formal definition are

$$\lim_{x\to\pm\infty} k = k \qquad \text{and} \qquad \lim_{x\to\pm\infty} \frac{1}{x} = 0. \tag{1}$$

We prove the second result in Example 1, and leave the first to Exercises 87 and 88.

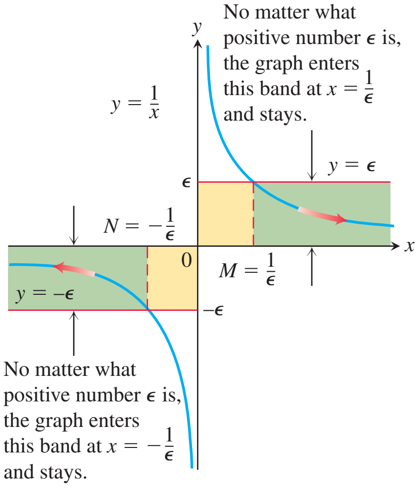

FIGURE 2.50 The geometry behind the argument in Example 1.

EXAMPLE 1 Show that

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$ **(b)** $\lim_{x\to-\infty} \frac{1}{x} = 0.$

Solution

(a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

$$x > M \qquad \Rightarrow \qquad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (Figure 2.50). This proves $\lim_{x\to\infty}(1/x) = 0$.

(b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \qquad \Rightarrow \qquad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$ (Figure 2.50). This proves $\lim_{x\to-\infty}(1/x) = 0$. ■

Limits at infinity have properties similar to those of finite limits.

THEOREM 12 All the Limit Laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

EXAMPLE 2 The properties in Theorem 12 are used to calculate limits in the same way as when x approaches a finite number c.

$$\textbf{(a)}\quad \lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} \qquad \text{Sum Rule}$$

$$= 5 + 0 = 5 \qquad \text{Known limits}$$

$$\textbf{(b)}\quad \lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$$

$$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} \qquad \text{Product Rule}$$

$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0 \qquad \text{Known limits}$$

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

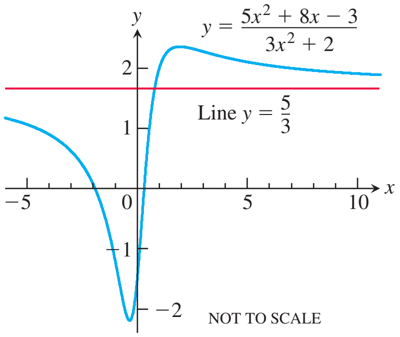

FIGURE 2.51 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

$$\textbf{(a)}\quad \lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)} \qquad \text{Divide numerator and denominator by } x^2.$$

$$= \frac{5+0-0}{3+0} = \frac{5}{3} \qquad \text{See Fig. 2.51.}$$

$$\textbf{(b)}\quad \lim_{x\to-\infty}\frac{11x+2}{2x^3-1} = \lim_{x\to-\infty}\frac{(11/x^2)+(2/x^3)}{2-(1/x^3)} \qquad \text{Divide numerator and denominator by } x^3.$$

$$= \frac{0+0}{2-0} = 0 \qquad \text{See Fig. 2.52.}$$

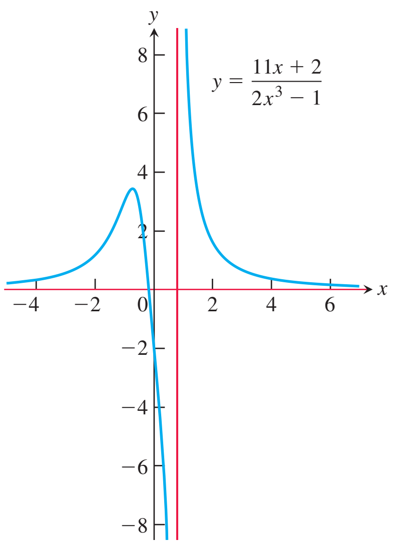

FIGURE 2.52 The graph of the function in Example 3b. The graph approaches the x-axis as $|x|$ increases.

Cases for which the degree of the numerator is greater than the degree of the denominator are illustrated in Examples 10 and 14.

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

Looking at $f(x) = 1/x$ (see Figure 2.49), we observe that the x-axis is an asymptote of the curve on the right because

$$\lim_{x\to\infty}\frac{1}{x} = 0$$

and on the left because

$$\lim_{x\to-\infty}\frac{1}{x} = 0.$$

We say that the x-axis is a *horizontal asymptote* of the graph of $f(x) = 1/x$.

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b.$$

The graph of the function

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

sketched in Figure 2.51 (Example 3a) has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x \to \infty} f(x) = \frac{5}{3} \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = \frac{5}{3}.$$

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

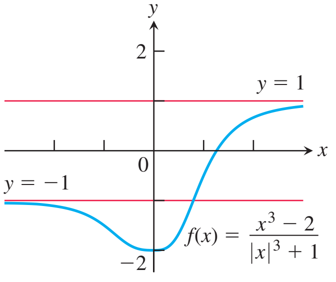

FIGURE 2.53 The graph of the function in Example 4 has two horizontal asymptotes.

Solution We calculate the limits as $x \to \pm\infty$.

For $x \geq 0$: $\displaystyle \lim_{x \to \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \to \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \to \infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$

For $x < 0$: $\displaystyle \lim_{x \to -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \to -\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x \to -\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.53. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x. ■

EXAMPLE 5 The x-axis (the line $y = 0$) is a horizontal asymptote of the graph of $y = e^x$ because

$$\lim_{x \to -\infty} e^x = 0.$$

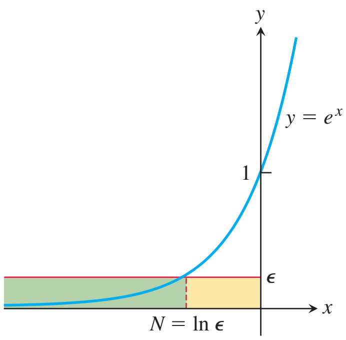

FIGURE 2.54 The graph of $y = e^x$ approaches the x-axis as $x \to -\infty$ (Example 5).

To see this, we use the definition of a limit as x approaches $-\infty$. So let $\epsilon > 0$ be given, but arbitrary. We must find a constant N such that for all x,

$$x < N \quad \Rightarrow \quad |e^x - 0| < \epsilon.$$

Now $|e^x - 0| = e^x$, so the condition that needs to be satisfied whenever $x < N$ is

$$e^x < \epsilon.$$

Let $x = N$ be the number where $e^x = \epsilon$. Since e^x is an increasing function, if $x < N$, then $e^x < \epsilon$. We find N by taking the natural logarithm of both sides of the equation $e^N = \epsilon$, so $N = \ln \epsilon$ (see Figure 2.54). With this value of N the condition is satisfied, and we conclude that $\lim_{x \to -\infty} e^x = 0$. ■

EXAMPLE 6 Find **(a)** $\displaystyle \lim_{x \to \infty} \sin(1/x)$ and **(b)** $\displaystyle \lim_{x \to \pm\infty} x \sin(1/x)$.

Solution

(a) We introduce the new variable $t = 1/x$. From Example 1, we know that $t \to 0^+$ as $x \to \infty$ (see Figure 2.49). Therefore,

$$\lim_{x \to \infty} \sin \frac{1}{x} = \lim_{t \to 0^+} \sin t = 0.$$

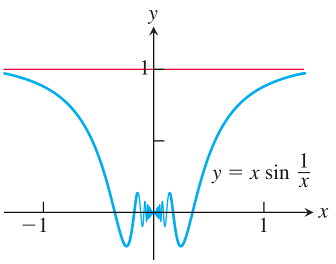

FIGURE 2.55 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 6b).

(b) We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{x\to-\infty} x \sin\frac{1}{x} = \lim_{t\to 0^-} \frac{\sin t}{t} = 1.$$

The graph is shown in Figure 2.55, and we see that the line $y = 1$ is a horizontal asymptote. ■

Likewise, we can investigate the behavior of $y = f(1/x)$ as $x \to 0$ by investigating $y = f(t)$ as $t \to \pm\infty$, where $t = 1/x$.

EXAMPLE 7 Find $\lim_{x\to 0^-} e^{1/x}$.

Solution We let $t = 1/x$. From Figure 2.49, we can see that $t \to -\infty$ as $x \to 0^-$. (We make this idea more precise further on.) Therefore,

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0 \qquad \text{Example 5}$$

(Figure 2.56). ■

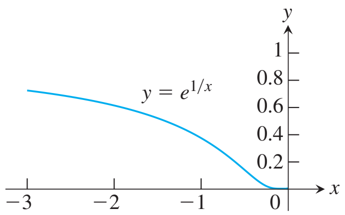

FIGURE 2.56 The graph of $y = e^{1/x}$ for $x < 0$ shows $\lim_{x\to 0^-} e^{1/x} = 0$ (Example 7).

The Sandwich Theorem also holds for limits as $x \to \pm\infty$. You must be sure, though, that the function whose limit you are trying to find stays between the bounding functions at very large values of x in magnitude consistent with whether $x \to \infty$ or $x \to -\infty$.

EXAMPLE 8 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x\to\pm\infty} |1/x| = 0$, we have $\lim_{x\to\pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.57).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

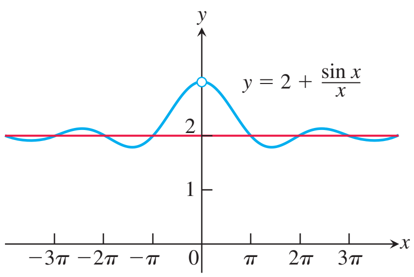

FIGURE 2.57 A curve may cross one of its asymptotes infinitely often (Example 8).

EXAMPLE 9 Find $\lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right)$.

Solution Both of the terms x and $\sqrt{x^2 + 16}$ approach infinity as $x \to \infty$, so what happens to the difference in the limit is unclear (we cannot subtract ∞ from ∞ because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic result:

$$\begin{aligned}
\lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right) &= \lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right)\frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}} \\
&= \lim_{x\to\infty}\frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} = \lim_{x\to\infty}\frac{-16}{x + \sqrt{x^2 + 16}}.
\end{aligned}$$

As $x \to \infty$, the denominator in this last expression becomes arbitrarily large, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x\to\infty} \frac{-16}{x + \sqrt{x^2 + 16}} = \lim_{x\to\infty} \frac{-\dfrac{16}{x}}{1 + \sqrt{\dfrac{x^2}{x^2} + \dfrac{16}{x^2}}} = \frac{0}{1 + \sqrt{1 + 0}} = 0.$$

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

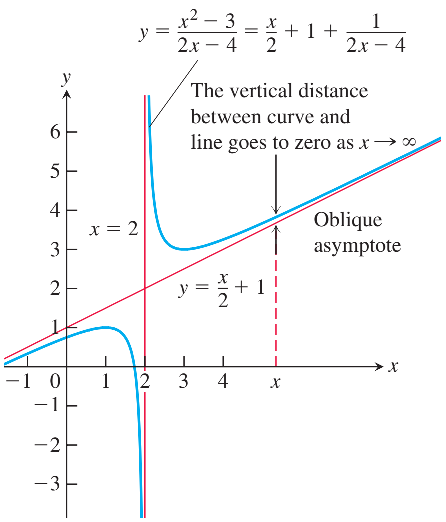

FIGURE 2.58 The graph of the function in Example 10 has an oblique asymptote.

EXAMPLE 10 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Figure 2.58.

Solution We are interested in the behavior as $x \to \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r}
\dfrac{x}{2} + 1 \\
2x - 4 \overline{) \, x^2 - 3} \\
\underline{x^2 - 2x} \\
2x - 3 \\
\underline{2x - 4} \\
1
\end{array}$$

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}.$$

As $x \to \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of f (Figure 2.58). The line $y = g(x)$ is an asymptote both to the right and to the left. The next subsection will confirm that the function $f(x)$ grows arbitrarily large in absolute value as $x \to 2$ (where the denominator is zero), as shown in the graph.

Notice in Example 10 that if the degree of the numerator in a rational function is greater than the degree of the denominator, then the limit as $|x|$ becomes large is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator.

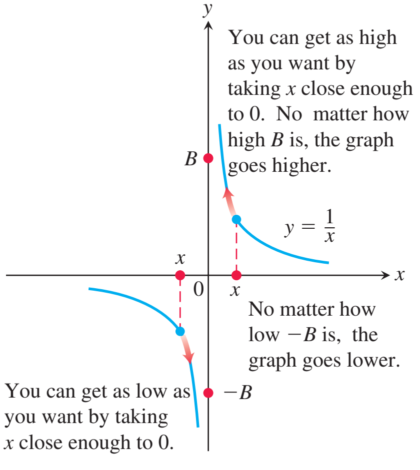

FIGURE 2.59 One-sided infinite limits:
$\lim_{x\to 0^+} \dfrac{1}{x} = \infty$ and $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty.$

Infinite Limits

Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Figure 2.59).

Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x \to 0^+} (1/x)$ *does not exist because $1/x$ becomes arbitrarily large and positive as $x \to 0^+$.*

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Figure 2.59.) We write

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative.*

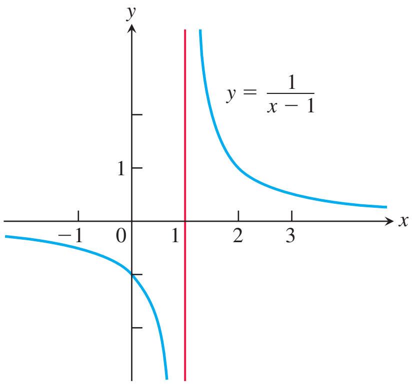

FIGURE 2.60 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 11).

EXAMPLE 11 Find $\lim_{x \to 1^+} \frac{1}{x - 1}$ and $\lim_{x \to 1^-} \frac{1}{x - 1}$.

Geometric Solution The graph of $y = 1/(x - 1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.60). Therefore, $y = 1/(x - 1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x \to 1^+} \frac{1}{x - 1} = \infty \qquad \text{and} \qquad \lim_{x \to 1^-} \frac{1}{x - 1} = -\infty.$$

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x - 1) \to 0^+$ and $1/(x - 1) \to \infty$. As $x \to 1^-$, we have $(x - 1) \to 0^-$ and $1/(x - 1) \to -\infty$. ■

EXAMPLE 12 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \qquad \text{as} \qquad x \to 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.61). This means that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x \to 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x \to 0} (1/x^2) = \infty$. ■

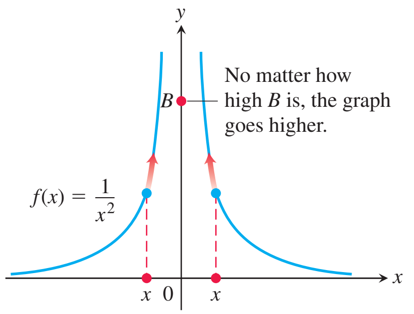

FIGURE 2.61 The graph of $f(x)$ in Example 12 approaches infinity as $x \to 0$.

EXAMPLE 13 These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

(a) $\lim_{x \to 2} \frac{(x - 2)^2}{x^2 - 4} = \lim_{x \to 2} \frac{(x - 2)^2}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{x - 2}{x + 2} = 0$

(b) $\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{1}{x + 2} = \frac{1}{4}$

(c) $\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$ 　　The values are negative for $x > 2$, x near 2.

(d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$ 　　The values are positive for $x < 2$, x near 2.

(e) $\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$ does not exist. 　　See parts (c) and (d).

(f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator. ■

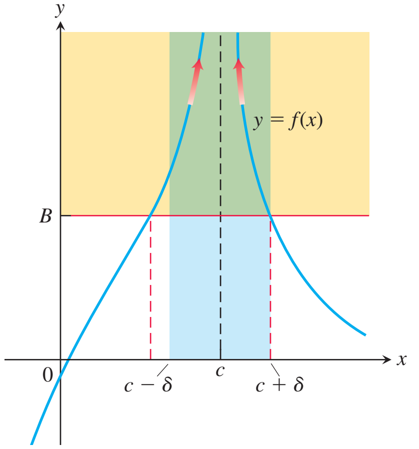

FIGURE 2.62 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

EXAMPLE 14 Find $\lim_{x\to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

Solution We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by x^2, the highest power of x in the denominator:

$$\begin{aligned}
\lim_{x\to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x\to -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= \lim_{x\to -\infty} \frac{2x^2(x-3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= -\infty, \qquad x^{-n} \to 0,\ x - 3 \to -\infty
\end{aligned}$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$. ■

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to c, the definitions of infinite limits require $f(x)$ to lie arbitrarily far from zero. Except for this change, the language is very similar to what we have seen before. Figures 2.62 and 2.63 accompany these definitions.

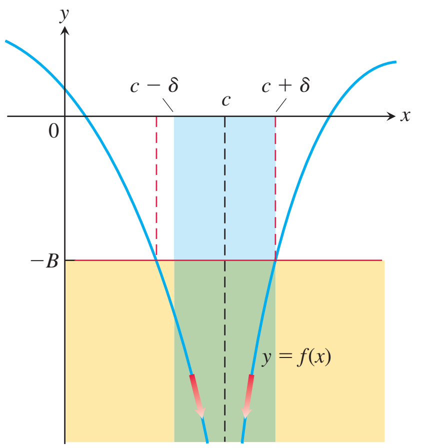

FIGURE 2.63 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITIONS

1. We say that **$f(x)$ approaches infinity as x approaches c**, and write
$$\lim_{x\to c} f(x) = \infty,$$
if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches minus infinity as x approaches c**, and write
$$\lim_{x\to c} f(x) = -\infty,$$
if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) < -B.$$

The precise definitions of one-sided infinite limits at c are similar and are stated in the exercises.