(c) $\displaystyle \lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$ — The values are negative for $x > 2$, x near 2.

(d) $\displaystyle \lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$ — The values are positive for $x < 2$, x near 2.

(e) $\displaystyle \lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$ does not exist. — See parts (c) and (d).

(f) $\displaystyle \lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator. ■

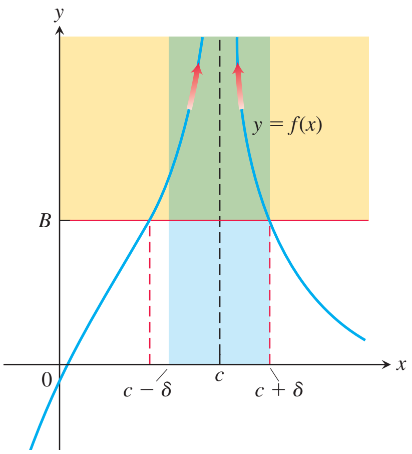


FIGURE 2.62 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

EXAMPLE 14 Find $\displaystyle \lim_{x\to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

Solution We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by x^2, the highest power of x in the denominator:

$$\begin{aligned}
\lim_{x\to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x\to -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= \lim_{x\to -\infty} \frac{2x^2(x-3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\
&= -\infty, \qquad x^{-n} \to 0,\ x - 3 \to -\infty
\end{aligned}$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$. ■

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to c, the definitions of infinite limits require $f(x)$ to lie arbitrarily far from zero. Except for this change, the language is very similar to what we have seen before. Figures 2.62 and 2.63 accompany these definitions.

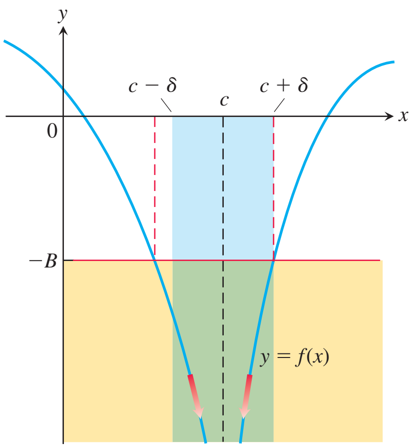


FIGURE 2.63 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITIONS

1. We say that **$f(x)$ approaches infinity as x approaches c**, and write

$$\lim_{x\to c} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches minus infinity as x approaches c**, and write

$$\lim_{x\to c} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) < -B.$$

The precise definitions of one-sided infinite limits at c are similar and are stated in the exercises.

EXAMPLE 15 Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \qquad \text{if and only if} \qquad x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Vertical Asymptotes

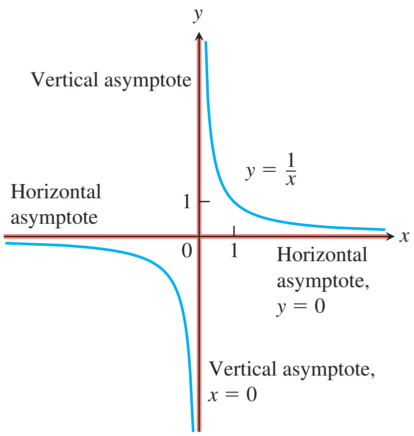


FIGURE 2.64 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

Notice that the distance between a point on the graph of $f(x) = 1/x$ and the y-axis approaches zero as the point moves vertically along the graph and away from the origin (Figure 2.64). The function $f(x) = 1/x$ is unbounded as x approaches 0 because

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

We say that the line $x = 0$ (the y-axis) is a *vertical asymptote* of the graph of $f(x) = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

EXAMPLE 16 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x + 3}{x + 2}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and the behavior as $x \to -2$, where the denominator is zero.

The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x + 2)$ into $(x + 3)$:

$$\begin{array}{r} 1 \\ x + 2 \overline{)\, x + 3} \\ \underline{x + 2} \\ 1 \end{array}$$

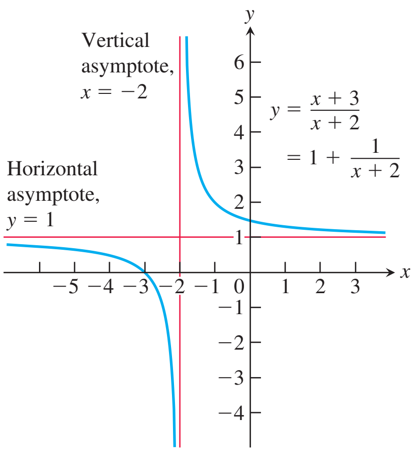


FIGURE 2.65 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 16.

This result enables us to rewrite y as:

$$y = 1 + \frac{1}{x+2}.$$

As $x \to \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \to -2$, the curve approaches the vertical asymptote $x = -2$. We see that the curve in question is the graph of $f(x) = 1/x$ shifted 1 unit up and 2 units left (Figure 2.65). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. ■

EXAMPLE 17 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.

(a) *The behavior as* $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.66). Notice that the curve approaches the x-axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) *The behavior as* $x \to \pm 2$. Since

$$\lim_{x\to 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at all other points. ■

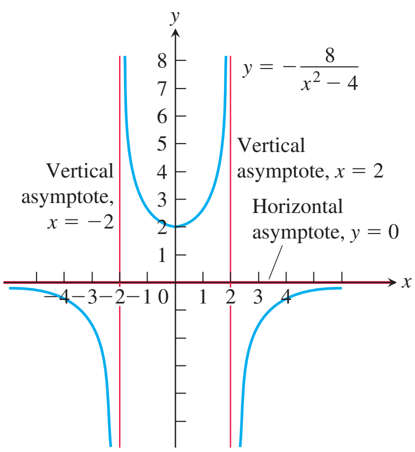


FIGURE 2.66 Graph of the function in Example 17. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

EXAMPLE 18 The graph of the natural logarithm function has the y-axis (the line $x = 0$) as a vertical asymptote. We see this from the graph sketched in Figure 2.67 (which is the reflection of the graph of the natural exponential function across the line $y = x$) and the fact that the x-axis is a horizontal asymptote of $y = e^x$ (Example 5). Thus,

$$\lim_{x\to 0^+} \ln x = -\infty.$$

The same result is true for $y = \log_a x$ whenever $a > 1$. ■

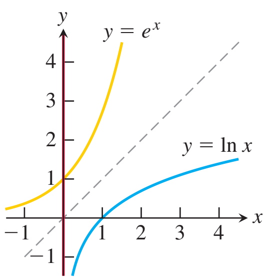


FIGURE 2.67 The line $x = 0$ is a vertical asymptote of the natural logarithm function (Example 18).

EXAMPLE 19 The curves

$$y = \sec x = \frac{1}{\cos x} \quad \text{and} \quad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Figure 2.68).

Dominant Terms

In Example 10 we saw that by long division we could rewrite the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

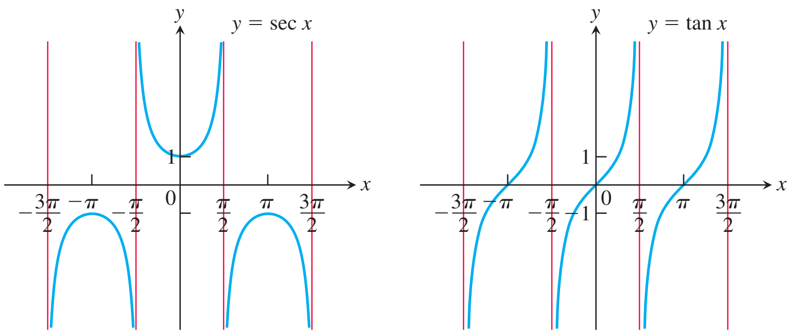


FIGURE 2.68 The graphs of sec x and tan x have infinitely many vertical asymptotes (Example 19).

as a linear function plus a remainder term:

$$f(x) = \left(\frac{x}{2} + 1\right) + \left(\frac{1}{2x - 4}\right).$$

This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \qquad \text{For } |x| \text{ large, } \frac{1}{2x-4} \text{ is near 0.}$$

$$f(x) \approx \frac{1}{2x - 4} \qquad \text{For x near 2, this term is very large in absolute value.}$$

If we want to know how f behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when $|x|$ is large and the contribution of $1/(2x - 4)$ to the total value of f is insignificant. It behaves like $1/(2x - 4)$ when x is so close to 2 that $1/(2x - 4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x is numerically large, and we say that $1/(2x - 4)$ dominates when x is near 2. **Dominant terms** like these help us predict a function's behavior.

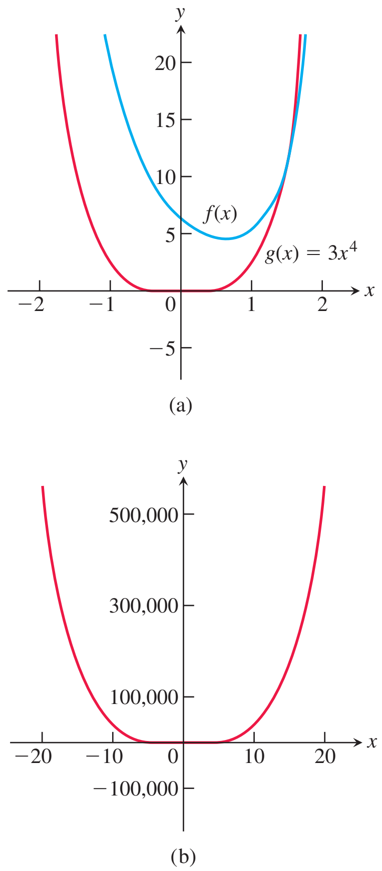


FIGURE 2.69 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 20).

EXAMPLE 20 Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x, they are virtually identical for $|x|$ very large, in the sense that their ratios approach 1 as $x \to \infty$ or $x \to -\infty$.

Solution The graphs of f and g behave quite differently near the origin (Figure 2.69a), but appear as virtually identical on a larger scale (Figure 2.69b).

We can test that the term $3x^4$ in f, represented graphically by g, dominates the polynomial f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x\to\pm\infty}\left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right) \\ &= 1,\end{aligned}$$

which means that f and g appear nearly identical when $|x|$ is large.

Summary

In this chapter we presented several important calculus ideas that are made meaningful and precise by the concept of the limit. These include the three ideas of the exact rate of change of a function, the slope of the graph of a function at a point, and the continuity of a function. The primary methods used for calculating limits of many functions are captured in the algebraic

Limit Laws of Theorem 1 and in the Sandwich Theorem, all of which are proved from the precise definition of the limit. We saw that these computational rules also apply to one-sided limits and to limits at infinity. Moreover, we can sometimes apply these rules when calculating limits of simple transcendental functions, as illustrated by our examples or in cases like the following:

$$\lim_{x\to 0}\frac{e^x-1}{e^{2x}-1}=\lim_{x\to 0}\frac{e^x-1}{(e^x-1)(e^x+1)}=\lim_{x\to 0}\frac{1}{e^x+1}=\frac{1}{1+1}=\frac{1}{2}.$$

However, calculating more complicated limits involving transcendental functions such as

$$\lim_{x\to 0}\frac{x}{e^{2x}-1},\qquad \lim_{x\to 0}\frac{\ln x}{x},\quad\text{and}\quad \lim_{x\to 0}\left(1+\frac{1}{x}\right)^x$$

requires more than simple algebraic techniques. The *derivative* is exactly the tool we need to calculate limits such as these (see Section 4.5), and this notion is the main subject of our next chapter.

Exercises 2.6

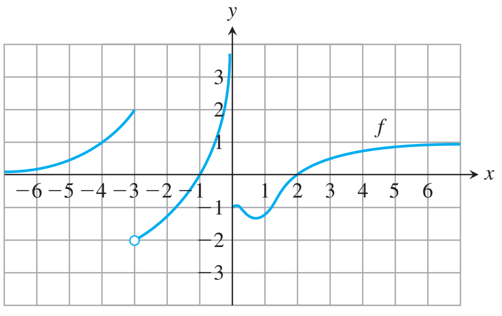
Finding Limits

1. For the function f whose graph is given, determine the following limits.

a. $\lim_{x\to 2} f(x)$ **b.** $\lim_{x\to -3^+} f(x)$ **c.** $\lim_{x\to -3^-} f(x)$

d. $\lim_{x\to -3} f(x)$ **e.** $\lim_{x\to 0^+} f(x)$ **f.** $\lim_{x\to 0^-} f(x)$

g. $\lim_{x\to 0} f(x)$ **h.** $\lim_{x\to \infty} f(x)$ **i.** $\lim_{x\to -\infty} f(x)$



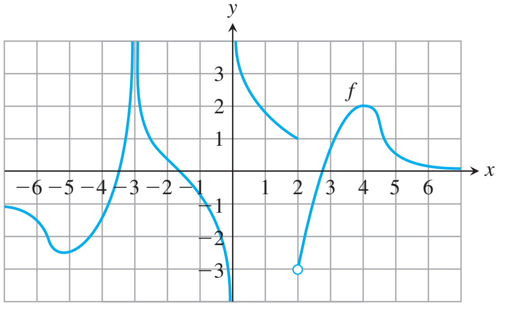
2. For the function f whose graph is given, determine the following limits.

a. $\lim_{x\to 4} f(x)$ **b.** $\lim_{x\to 2^+} f(x)$ **c.** $\lim_{x\to 2^-} f(x)$

d. $\lim_{x\to 2} f(x)$ **e.** $\lim_{x\to -3^+} f(x)$ **f.** $\lim_{x\to -3^-} f(x)$

g. $\lim_{x\to -3} f(x)$ **h.** $\lim_{x\to 0^+} f(x)$ **i.** $\lim_{x\to 0^-} f(x)$

j. $\lim_{x\to 0} f(x)$ **k.** $\lim_{x\to \infty} f(x)$ **l.** $\lim_{x\to -\infty} f(x)$



In Exercises 3–8, find the limit of each function **(a)** as $x\to\infty$ and **(b)** as $x\to-\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

3. $f(x)=\frac{2}{x}-3$ **4.** $f(x)=\pi-\frac{2}{x^2}$

5. $g(x)=\frac{1}{2+(1/x)}$ **6.** $g(x)=\frac{1}{8-(5/x^2)}$

7. $h(x)=\frac{-5+(7/x)}{3-(1/x^2)}$ **8.** $h(x)=\frac{3-(2/x)}{4+(\sqrt{2}/x^2)}$

Find the limits in Exercises 9–12.

9. $\lim_{x\to\infty}\frac{\sin 2x}{x}$ **10.** $\lim_{\theta\to-\infty}\frac{\cos\theta}{3\theta}$

11. $\lim_{t\to-\infty}\frac{2-t+\sin t}{t+\cos t}$ **12.** $\lim_{r\to\infty}\frac{r+\sin r}{2r+7-5\sin r}$

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x\to\infty$ and **(b)** as $x\to-\infty$.

13. $f(x)=\frac{2x+3}{5x+7}$ **14.** $f(x)=\frac{2x^3+7}{x^3-x^2+x+7}$

15. $f(x)=\frac{x+1}{x^2+3}$ **16.** $f(x)=\frac{3x+7}{x^2-2}$

17. $h(x)=\frac{7x^3}{x^3-3x^2+6x}$ **18.** $h(x)=\frac{9x^4+x}{2x^4+5x^2-x+6}$

19. $g(x)=\frac{10x^5+x^4+31}{x^6}$ **20.** $g(x)=\frac{x^3+7x^2-2}{x^2-x+1}$

21. $f(x)=\frac{3x^7+5x^2-1}{6x^3-7x+3}$ **22.** $h(x)=\frac{5x^8-2x^3+9}{3+x-4x^5}$

Limits as $x\to\infty$ or $x\to-\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36.

23. $\lim_{x\to\infty}\sqrt{\frac{8x^2-3}{2x^2+x}}$ **24.** $\lim_{x\to-\infty}\left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3}$

25. $\lim_{x\to-\infty}\left(\frac{1-x^3}{x^2+7x}\right)^5$

26. $\lim_{x\to\infty}\sqrt{\frac{x^2-5x}{x^3+x-2}}$

27. $\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7}$

28. $\lim_{x\to\infty}\frac{2+\sqrt{x}}{2-\sqrt{x}}$

29. $\lim_{x\to-\infty}\frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

30. $\lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$

31. $\lim_{x\to\infty}\frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}$

32. $\lim_{x\to-\infty}\frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$

33. $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x+1}$

34. $\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1}$

35. $\lim_{x\to\infty}\frac{x-3}{\sqrt{4x^2+25}}$

36. $\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}$

Infinite Limits

Find the limits in Exercises 37–48.

37. $\lim_{x\to0^+}\frac{1}{3x}$

38. $\lim_{x\to0^-}\frac{5}{2x}$

39. $\lim_{x\to2^-}\frac{3}{x-2}$

40. $\lim_{x\to3^+}\frac{1}{x-3}$

41. $\lim_{x\to-8^+}\frac{2x}{x+8}$

42. $\lim_{x\to-5^-}\frac{3x}{2x+10}$

43. $\lim_{x\to7}\frac{4}{(x-7)^2}$

44. $\lim_{x\to0}\frac{-1}{x^2(x+1)}$

45. **a.** $\lim_{x\to0^+}\frac{2}{3x^{1/3}}$ **b.** $\lim_{x\to0^-}\frac{2}{3x^{1/3}}$

46. **a.** $\lim_{x\to0^+}\frac{2}{x^{1/5}}$ **b.** $\lim_{x\to0^-}\frac{2}{x^{1/5}}$

47. $\lim_{x\to0}\frac{4}{x^{2/5}}$

48. $\lim_{x\to0}\frac{1}{x^{2/3}}$

Find the limits in Exercises 49–52.

49. $\lim_{x\to(\pi/2)^-}\tan x$

50. $\lim_{x\to(-\pi/2)^+}\sec x$

51. $\lim_{\theta\to0^-}(1+\csc\theta)$

52. $\lim_{\theta\to0}(2-\cot\theta)$

Find the limits in Exercises 53–58.

53. $\lim\frac{1}{x^2-4}$ as

a. $x\to2^+$ **b.** $x\to2^-$
c. $x\to-2^+$ **d.** $x\to-2^-$

54. $\lim\frac{x}{x^2-1}$ as

a. $x\to1^+$ **b.** $x\to1^-$
c. $x\to-1^+$ **d.** $x\to-1^-$

55. $\lim\left(\frac{x^2}{2}-\frac{1}{x}\right)$ as

a. $x\to0^+$ **b.** $x\to0^-$
c. $x\to\sqrt[3]{2}$ **d.** $x\to-1$

56. $\lim\frac{x^2-1}{2x+4}$ as

a. $x\to-2^+$ **b.** $x\to-2^-$
c. $x\to1^+$ **d.** $x\to0^-$

57. $\lim\frac{x^2-3x+2}{x^3-2x^2}$ as

a. $x\to0^+$ **b.** $x\to2^+$
c. $x\to2^-$ **d.** $x\to2$
e. What, if anything, can be said about the limit as $x\to0$?

58. $\lim\frac{x^2-3x+2}{x^3-4x}$ as

a. $x\to2^+$ **b.** $x\to-2^+$
c. $x\to0^-$ **d.** $x\to1^+$
e. What, if anything, can be said about the limit as $x\to0$?

Find the limits in Exercises 59–62.

59. $\lim\left(2-\frac{3}{t^{1/3}}\right)$ as

a. $t\to0^+$ **b.** $t\to0^-$

60. $\lim\left(\frac{1}{t^{3/5}}+7\right)$ as

a. $t\to0^+$ **b.** $t\to0^-$

61. $\lim\left(\frac{1}{x^{2/3}}+\frac{2}{(x-1)^{2/3}}\right)$ as

a. $x\to0^+$ **b.** $x\to0^-$
c. $x\to1^+$ **d.** $x\to1^-$

62. $\lim\left(\frac{1}{x^{1/3}}-\frac{1}{(x-1)^{4/3}}\right)$ as

a. $x\to0^+$ **b.** $x\to0^-$
c. $x\to1^+$ **d.** $x\to1^-$

Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

63. $y=\frac{1}{x-1}$

64. $y=\frac{1}{x+1}$

65. $y=\frac{1}{2x+4}$

66. $y=\frac{-3}{x-3}$

67. $y=\frac{x+3}{x+2}$

68. $y=\frac{2x}{x+1}$

Inventing Graphs and Functions

In Exercises 69–72, sketch the graph of a function $y=f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

69. $f(0)=0$, $f(1)=2$, $f(-1)=-2$, $\lim_{x\to-\infty}f(x)=-1$, and $\lim_{x\to\infty}f(x)=1$

70. $f(0)=0$, $\lim_{x\to\pm\infty}f(x)=0$, $\lim_{x\to0^+}f(x)=2$, and $\lim_{x\to0^-}f(x)=-2$

71. $f(0)=0$, $\lim_{x\to\pm\infty}f(x)=0$, $\lim_{x\to1^-}f(x)=\lim_{x\to-1^+}f(x)=\infty$, $\lim_{x\to1^+}f(x)=-\infty$, and $\lim_{x\to-1^-}f(x)=-\infty$

72. $f(2)=1$, $f(-1)=0$, $\lim_{x\to\infty}f(x)=0$, $\lim_{x\to0^+}f(x)=\infty$, $\lim_{x\to0^-}f(x)=-\infty$, and $\lim_{x\to-\infty}f(x)=1$

In Exercises 73–76, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

73. $\lim_{x\to\pm\infty} f(x) = 0$, $\lim_{x\to 2^-} f(x) = \infty$, and $\lim_{x\to 2^+} f(x) = \infty$

74. $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$

75. $\lim_{x\to-\infty} h(x) = -1$, $\lim_{x\to\infty} h(x) = 1$, $\lim_{x\to 0^-} h(x) = -1$, and $\lim_{x\to 0^+} h(x) = 1$

76. $\lim_{x\to\pm\infty} k(x) = 1$, $\lim_{x\to 1^-} k(x) = \infty$, and $\lim_{x\to 1^+} k(x) = -\infty$

77. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x\to\infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x\to-\infty} (f(x)/g(x))$? Give reasons for your answer.

78. Suppose that $f(x)$ and $g(x)$ are polynomials in x. Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

79. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

Finding Limits of Differences When $x \to \pm\infty$

Find the limits in Exercises 80–86.

80. $\lim_{x\to\infty} \left(\sqrt{x+9} - \sqrt{x+4}\right)$

81. $\lim_{x\to\infty} \left(\sqrt{x^2+25} - \sqrt{x^2-1}\right)$

82. $\lim_{x\to-\infty} \left(\sqrt{x^2+3} + x\right)$

83. $\lim_{x\to-\infty} \left(2x + \sqrt{4x^2+3x-2}\right)$

84. $\lim_{x\to\infty} \left(\sqrt{9x^2-x} - 3x\right)$

85. $\lim_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right)$

86. $\lim_{x\to\infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)$

Using the Formal Definitions

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 87 and 88.

87. If f has the constant value $f(x) = k$, then $\lim_{x\to\infty} f(x) = k$.

88. If f has the constant value $f(x) = k$, then $\lim_{x\to-\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 89–92.

89. $\lim_{x\to 0} \dfrac{-1}{x^2} = -\infty$

90. $\lim_{x\to 0} \dfrac{1}{|x|} = \infty$

91. $\lim_{x\to 3} \dfrac{-2}{(x-3)^2} = -\infty$

92. $\lim_{x\to -5} \dfrac{1}{(x+5)^2} = \infty$

93. Here is the definition of **infinite right-hand limit**.

> We say that $f(x)$ approaches infinity as x approaches c from the right, and write
>
> $$\lim_{x\to c^+} f(x) = \infty,$$
>
> if, for every positive real number B, there exists a corresponding number $\delta > 0$ such that for all x
>
> $$c < x < c + \delta \quad \Rightarrow \quad f(x) > B.$$

Modify the definition to cover the following cases.

a. $\lim_{x\to c^-} f(x) = \infty$

b. $\lim_{x\to c^+} f(x) = -\infty$

c. $\lim_{x\to c^-} f(x) = -\infty$

Use the formal definitions from Exercise 93 to prove the limit statements in Exercises 94–98.

94. $\lim_{x\to 0^+} \dfrac{1}{x} = \infty$

95. $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty$

96. $\lim_{x\to 2^-} \dfrac{1}{x-2} = -\infty$

97. $\lim_{x\to 2^+} \dfrac{1}{x-2} = \infty$

98. $\lim_{x\to 1^-} \dfrac{1}{1-x^2} = \infty$

Oblique Asymptotes

Graph the rational functions in Exercises 99–104. Include the graphs and equations of the asymptotes.

99. $y = \dfrac{x^2}{x-1}$

100. $y = \dfrac{x^2+1}{x-1}$

101. $y = \dfrac{x^2-4}{x-1}$

102. $y = \dfrac{x^2-1}{2x+4}$

103. $y = \dfrac{x^2-1}{x}$

104. $y = \dfrac{x^3+1}{x^2}$

Additional Graphing Exercises

T Graph the curves in Exercises 105–108. Explain the relationship between the curve's formula and what you see.

105. $y = \dfrac{x}{\sqrt{4-x^2}}$

106. $y = \dfrac{-1}{\sqrt{4-x^2}}$

107. $y = x^{2/3} + \dfrac{1}{x^{1/3}}$

108. $y = \sin\left(\dfrac{\pi}{x^2+1}\right)$

T Graph the functions in Exercises 109 and 110. Then answer the following questions.

a. How does the graph behave as $x \to 0^+$?

b. How does the graph behave as $x \to \pm\infty$?

c. How does the graph behave near $x = 1$ and $x = -1$?

Give reasons for your answers.

109. $y = \dfrac{3}{2}\left(x - \dfrac{1}{x}\right)^{2/3}$

110. $y = \dfrac{3}{2}\left(\dfrac{x}{x-1}\right)^{2/3}$

Chapter 2 Questions to Guide Your Review

1. What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?
2. What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?
3. Give an informal or intuitive definition of the limit
$$\lim_{x \to c} f(x) = L.$$
Why is the definition "informal"? Give examples.
4. Does the existence and value of the limit of a function $f(x)$ as x approaches c ever depend on what happens at $x = c$? Explain and give examples.
5. What function behaviors might occur for which the limit may fail to exist? Give examples.
6. What theorems are available for calculating limits? Give examples of how the theorems are used.
7. How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.
8. What is the value of $\lim_{\theta \to 0} ((\sin\theta)/\theta)$? Does it matter whether θ is measured in degrees or radians? Explain.
9. What exactly does $\lim_{x \to c} f(x) = L$ mean? Give an example in which you find a $\delta > 0$ for a given f, L, c, and $\epsilon > 0$ in the precise definition of limit.
10. Give precise definitions of the following statements.
    a. $\lim_{x \to 2^-} f(x) = 5$ **b.** $\lim_{x \to 2^+} f(x) = 5$
    c. $\lim_{x \to 2} f(x) = \infty$ **d.** $\lim_{x \to 2} f(x) = -\infty$
11. What conditions must be satisfied by a function if it is to be continuous at an interior point of its domain? At an endpoint?
12. How can looking at the graph of a function help you tell where the function is continuous?
13. What does it mean for a function to be right-continuous at a point? Left-continuous? How are continuity and one-sided continuity related?
14. What does it mean for a function to be continuous on an interval? Give examples to illustrate the fact that a function that is not continuous on its entire domain may still be continuous on selected intervals within the domain.
15. What are the basic types of discontinuity? Give an example of each. What is a removable discontinuity? Give an example.
16. What does it mean for a function to have the Intermediate Value Property? What conditions guarantee that a function has this property over an interval? What are the consequences for graphing and solving the equation $f(x) = 0$?
17. Under what circumstances can you extend a function $f(x)$ to be continuous at a point $x = c$? Give an example.
18. What exactly do $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to -\infty} f(x) = L$ mean? Give examples.
19. What are $\lim_{x \to \pm\infty} k$ (k a constant) and $\lim_{x \to \pm\infty} (1/x)$? How do you extend these results to other functions? Give examples.
20. How do you find the limit of a rational function as $x \to \pm\infty$? Give examples.
21. What are horizontal and vertical asymptotes? Give examples.

Chapter 2 Practice Exercises

Limits and Continuity

1. Graph the function
$$f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$
Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1, 0,$ and 1. Are any of the discontinuities removable? Explain.
2. Repeat the instructions of Exercise 1 for
$$f(x) = \begin{cases} 0, & x \le -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$
3. Suppose that $f(t)$ and $f(t)$ are defined for all t and that $\lim_{t \to t_0} f(t) = -7$ and $\lim_{t \to t_0} g(t) = 0$. Find the limit as $t \to t_0$ of the following functions.
    a. $3f(t)$ **b.** $(f(t))^2$
    c. $f(t) \cdot g(t)$ **d.** $\dfrac{f(t)}{g(t) - 7}$
    e. $\cos(g(t))$ **f.** $|f(t)|$
    g. $f(t) + g(t)$ **h.** $1/f(t)$
4. Suppose the functions $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x \to 0} f(x) = 1/2$ and $\lim_{x \to 0} g(x) = \sqrt{2}$. Find the limits as $x \to 0$ of the following functions.
    a. $-g(x)$ **b.** $g(x) \cdot f(x)$
    c. $f(x) + g(x)$ **d.** $1/f(x)$
    e. $x + f(x)$ **f.** $\dfrac{f(x) \cdot \cos x}{x - 1}$