3.6 Row space and column space.

Definition: A is $m \times n$ matrix

Row space of A

the subspace of $R^{1 \times n}$ spanned by row vectors A.

Column space of A

the subspace of R^{m} spanned by column vectors of A.

ex let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}$

Sol:

Row space of A = span$((1,0,0),(0,1,0))$

$= \alpha_1(1,0,0) + \alpha_2(0,1,0) = \{(\alpha_1,\alpha_2,0) ; \alpha_1,\alpha_2 \in R\} \longrightarrow$ subspace of $R^{1\times 3}$

Column space of A = span$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)$

$= \alpha_1\begin{bmatrix}1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\end{bmatrix} + \alpha_3\begin{bmatrix}0\\0\end{bmatrix}$

$= \left\{\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} ; \alpha_1,\alpha_2 \in R\right\} = R^2$

subspace of R^2

B is row equivalent to A

B can be formed from A by a finite sequence of row operations.

ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow$

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* **Thm** Row Equivalent matrices have the same row space.

Rank of $A_{m\times n}$

The dimension of the row space of A.

Thm Consistency theorem of linear System

* A linear System $Ax = b$ is consistent if and only if (b) is in the column space of A.

* An $n\times n$ matrix A is non-singular (invertible) if and only if the column vectors of A form a basis for R^n.

* Ex

Thm

- Nullity of $A_{m\times n}$:-
  the dimension of the Nullspace of A $(N(A))$.

- $A_{m\times n}$:-
  $$\text{Rank}(A) + \text{Nullity}(A) = n$$
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- dimension of Row space of A = dimension of Column space of A.

- $\dim(\text{Row space of } A) = \dim(\text{column space of } A)$
- $\dim(\text{Row space of } A) = \dim(\text{column space of } A^T)$

ex let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$

① find a <u>basis</u> of Row space of A?

sol: Find Row echelon form of A :-

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{REF of } A}$$

a basis for row space of A = $\{(1,2,-1,1), (0,0,1,2)\}$

② Rank (A) = 2

③ a basis for N(A) ?

(a basis for nullspace of A).

sol:

find N(A) ?? $\rightarrow$ $Ax = 0$ $\rightarrow$ find x

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 2 & 4 & -3 & 0 & 0 \\ 1 & 2 & 1 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables, x_2, x_4 $\rightarrow$ $\boxed{\begin{array}{l} x_2 = \alpha \\ x_4 = \beta \end{array}}$

$x_3 + 2x_4 = 0$

$x_3 + 2\beta = 0 \rightarrow \boxed{x_3 = -2\beta}$

$x_1 + 2x_2 - x_3 + x_4 = 0$

$x_1 + 2\alpha - (-2\beta) + \beta = 0$

$\boxed{x_1 = -2\alpha - 3\beta}$

$$N(A) = \left\{ \begin{bmatrix} -2\alpha - 3\beta \\ \alpha \\ -2\beta \\ \beta \end{bmatrix}, \ \alpha, \beta \in R \right\}$$

$$N(A) = \left\{ \begin{bmatrix} -2\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3\beta \\ 0 \\ -2\beta \\ \beta \end{bmatrix} \right\} = \left\{ \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right.$$

$, \alpha, \beta \in R$

a basis for $N(A) = \{ (-2, 1, 0, 0)^T, (3, 0, -2, 1)^T \}$

OR $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

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$\text{Nullity}(A) = 2$

<u>OR</u> $\text{Rank}(A) + \text{Nullity}(A) = n$

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$2 + \text{Nullity}(A) = 4 \rightarrow \text{Nullity}(A) = 2$

max: $\text{Rank}(A) = 3$

min: $\text{Nullity} = 1 \rightsquigarrow 4 - 3 = 1$

non-singular $n \times n$ matrix $\Leftrightarrow \text{Rank}(A) = n$

$\text{Nullity} = 0$

5 find a basis for column space of A.

<u>Sol:</u>

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{R.E.F of } A$$

a basis for column space of A =

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\} = \left\{ (1,2,1)^T, (-1,-3,1)^T \right\}$$

6 dim of column space of A =

dim of Row space of A = Rank = 2

Note

A, u satisfy the same dependency relation:

$u_2 = 2u_1 \Rightarrow a_2 = 2a_1$

$u_4 = 3u_1 + 2u_3 \longrightarrow a_4 = 3a_1 + 2a_3$.

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ex $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}$

$\rightarrow$ The Row echelon form of A:

$u = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Sol:

a basis for Row space of $A = \{(1,-2,1,1,2)$, $(0,1,1,3,0), (0,0,0,0,1)\}$

a basis for column space of $= \{(1,-1,0,1)^T$, $(-2,3,1,2)^T, (2,-2,4,5)^T\}$ $\dim$ column sp 3

Rank $= 3$, Nullity $(A) = 2$, $\dim N(A) = 2$, ex.

Find the dimension of the subspace of $\mathbb{R}^4$ spanned by :-

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad x_2 \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix}$$

sol:

Subspace span(x_1, x_2, x_3, x_4) =

column space of matrix $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

$\Rightarrow$ Row echelon form of A

$$\Rightarrow U = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for column space of A = $\{ (1, 2, -1, 0)^T, (2, 5, -3, 2)^T \}$

$\rightarrow$ dim column space of $A = 2$

$\rightarrow$ dim span$(x_1, x_2, x_3, x_4) = 2$.