chapter 5 : x

chapter 6 : Eigen values

Def:
let A be an $n \times n$ matrix, (square matrix).
a scalar (λ) is said to be eigen value or characteristic value of A if there exist a non-zero vector (X) such that $AX = \lambda X$.
The vector (X) is said to be an eigenvector or characteristic vector belonging to "λ"

$$\underbrace{A}_{n \times n \text{ matrix}} \underbrace{X}_{\text{eigen vector of } A} = \underbrace{\lambda}_{\text{eigen value of } A} X$$

ex let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find λ?

sol: $AX = \lambda X$

$$AX = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda X$$

$\longrightarrow \lambda = 3$.

let A be an $n \times n$ matrix, λ scalar.
the following statments are equivalent:

ⓐ λ is an eigen value of A
ⓑ $(A - \lambda I)X = 0$, has a nontrivial solution
ⓒ $N(A - \lambda I) \neq \{0\}$
ⓓ $A - \lambda I$ is singular.
ⓔ $\det(A - \lambda I) = 0$.

$AX = \lambda X$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0$, $X \neq 0$

to find (λ) eigen value.

$\Rightarrow \det(A - \lambda I) = 0$

to find (X) eigen vector

$\Rightarrow (A - \lambda I)0 = 0$

ex let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

find ① the eigen values of A

② the corresponding eigen vectors of A

Sol: ① find eigen values : $\rightarrow \lambda$

$$\det(A - \lambda I) = 0$$

$$\det\left[\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] = 0$$

$$\det \begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

نطرح λ من main diagonal of A

$$(4-\lambda)(1-\lambda) - -2 = 0$$

$$4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \longrightarrow (\lambda - 3)(\lambda - 2) = 0$$

$\rightarrow \lambda = 2, 3$. $\rightarrow$ eigen values : 2, 3.

② eigen vectors corresponding to λ

$\boxed{\lambda = 2}$ $(A - \lambda I) X = 0$.

$$\left[A - \lambda I \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right] \rightarrow \left[\begin{array}{cc|c} 4-\lambda & -2 & 0 \\ 1 & 1-\lambda & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 4-2 & -2 & 0 \\ 1 & 1-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\rightarrow$ infinitly many sol. [illegible]

[illegible] eigen vector.

x_2: free $\longrightarrow$ $\boxed{x_2 = \alpha}$

$x_1 - x_2 = 0 \rightarrow x_1 - \alpha = 0 \rightarrow \boxed{x_1 = \alpha}$

$= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \right\} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \alpha \in R \right\}$

eigen vector corresponding to $\lambda = 2$

eigen space : the set of all eigenvectors corresponding to λ.

$\left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \alpha \in R \right\}$

basis of the eigen space $= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, dim (eigenspace corr. to $\lambda = 2$) $= 1$

$\boxed{\lambda = 3}$ find corresponding eigen vectors to $\lambda = 3$

sol: $(A - \lambda I) X = 0 \Rightarrow \left[A - \lambda I \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right] \rightarrow$

$\rightarrow \left[\begin{array}{cc|c} 4-3 & -2 & 0 \\ 1 & 1-3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right] \rightarrow$

$\rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \boxed{x_2 = \alpha}$

$x_1 - 2x_2 = 0$

$x_1 - 2\alpha = 0 \rightarrow \boxed{x_1 = 2\alpha}$

eigen space: (eigen vectors) $= \left\{ \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \alpha \in R \right\}$

ex H.W

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \rightarrow \text{eigen value } \lambda = -3, 4$$

eigen vector: $\lambda = -3 \rightarrow \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\lambda = 4 \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

ex let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

Find eigen values of the corresponding eigen spaces.

Sol: find eigen values : $\rightarrow \lambda$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{bmatrix} = 0$$

$$2-\lambda \begin{vmatrix} -2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} + 3 \begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & -2-\lambda \\ 1 & -3 \end{vmatrix} = 0$$

$$(2-\lambda)\left[(-2-\lambda)(2-\lambda) + 3\right] + 3(2-\lambda-1) + 1(-3-(-2-\lambda)) = 0$$

$$(2-\lambda)\left[-4 + 2\lambda - 2\lambda + \lambda^2 + 3\right] + 3(1-\lambda) + (-1+\lambda) = 0$$

$$(2-\lambda)(\lambda^2 - 1) - 3(\lambda - 1) + (\lambda - 1) = 0$$

$$(2-\lambda)(\lambda-1)(\lambda+1) - 3(\lambda-1) + (\lambda-1) = 0$$

$$(\lambda-1)\left[(2-\lambda)(\lambda+1) - 3 + 1\right] = 0 \rightarrow (\lambda-1)\left[2\lambda + \cancel{2} - \lambda^2 - \lambda - \cancel{2}\right]$$

$$(\lambda-1)(-\lambda^2+\lambda) = 0$$

$$(\lambda-1)\,\lambda(-\lambda+1) = 0$$

$\lambda=1$ $\lambda=0$ $\lambda=1$

$\lambda = 0, 1, 1$

eigen spaces:

$\boxed{\lambda=0}$ $[A-\lambda I \mid 0]$

$$\to \left[\begin{array}{ccc|c} 2-0 & -3 & 1 & 0 \\ 1 & -2-0 & 1 & 0 \\ 1 & -3 & 2-0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \boxed{x_3=\alpha}$$

$x_2 - x_3 = 0 \to x_2 - \alpha = 0$

$\boxed{x_2 = \alpha}$

$x_1 - x_3 = 0 \to x_1 - \alpha = 0$

$\to \boxed{x_1 = \alpha}$

eigen space corresponding to $\lambda = 0$:-

$$\left\{ \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix}, \alpha \in R \right\} = \left\{ (\alpha,\alpha,\alpha)^T, \alpha \in R \right\}$$

$$= \left\{ \alpha(1,1,1)^T, \alpha \in R \right\}$$

$\boxed{\lambda=1}$ $(A-\lambda I \mid 0]$

$$\left[\begin{array}{ccc|c} 2-1 & -3 & 1 & 0 \\ 1 & -2-1 & 1 & 0 \\ 1 & -3 & 2-1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \boxed{\begin{matrix} x_2 = \alpha \\ x_3 = \beta \end{matrix}}, \quad \alpha, \beta \in R.$$

$$x_1 - 3x_2 + x_3 = 0 \rightarrow x_1 - 3\alpha + \beta = 0$$

$$\rightarrow \boxed{x_1 = 3\alpha - \beta}$$

eigen space corresponding to $\lambda = 1$

$$\left\{ \begin{bmatrix} 3\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} , \alpha, \beta \in R \right\} = \left\{ \begin{bmatrix} 3\alpha \\ \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} -\beta \\ 0 \\ \beta \end{bmatrix} \right\}$$

$$= \left\{ \alpha \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \alpha, \beta \in R \right\}.$$

Thm let A $n \times n$ matrix, $\lambda_1, \lambda_2, \dots, \lambda_n$ eigen values of A

① $\det(A) = \lambda_1 * \lambda_2 * \lambda_3 \cdots * \lambda_n$.

② $\text{trace}(A) = tr(A) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$

مجموع ال eigenvalues

مجموع عناصر القطر الرئيسي

ex $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

find $tr(A)$, find trace(A).

/sol: $tr(A) = 1 + -1 = 0$

ex let $A = \begin{bmatrix} x & 3 \\ y & 2 \end{bmatrix}$

and the characteristic polynomial

is $P(\lambda) = \lambda^2 - 5\lambda - 6$, find x, y.

Sol:

characteristic polynomial: $\rightarrow$

$\underbrace{\det(A - \lambda I)} = P(\lambda)$

$\Rightarrow$ let $P(\lambda) = 0$ to find.

$P(\lambda) = 0$

$\lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda - 6)(\lambda + 1) = 0$

$\rightarrow \lambda = 6, -1$.

⊛ $\det(A) = \lambda_1 * \lambda_2$

$2x - 3y = 6 * -1$

$\boxed{2x - 3y = -6}$

* $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} \Rightarrow 6 + -1 = x + 2$

$\Rightarrow 5 = x + 2 \rightarrow \boxed{x = 3}$

$2 \cdot 3 - 3y = -6 \rightarrow 6 - 3y = -6 \rightarrow 3y = 12$

$\rightarrow \boxed{y = 4}$

Similar Matrices

A matrix (B) is said to be similar to a matrix (A) if there exists a nonsingular matrix S, such that

$$B = S^{-1} A S.$$

Theorem

Similar Matrices have the same characteristic polynomial $\rightarrow$ the same eigen values $\rightarrow$ the same determinant.

ex $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, find B

$\rightarrow$ B is similar to A

Sol: $B = S^{-1} A S$.

$$B = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.$$

note If one of the eigen values is zero then $\rightarrow$ $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = 0$

$\rightarrow$ A is singular.

ex

diagonal matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \longrightarrow \text{eigen values:}$$

$$\det(A - \lambda I) = 0$$

$$(1-\lambda)(3-\lambda)(-4-\lambda) = 0$$

$$\lambda = 1, 3, -4. \longrightarrow$$

ex triangular matrix (upper or lower).

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 8 \\ 0 & 0 & -4 \end{bmatrix} \longrightarrow \text{eigen values of } A$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 5 & 7 \\ 0 & 3-\lambda & 8 \\ 0 & 0 & -4-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(3-\lambda)(-4-\lambda) = 0$$

$$\rightarrow \lambda = 1, 3, -4.$$

<u>Note</u> If A is an $n \times n$ triangular matrix, then the eigen values of A are the elements of the main diagonal of A.

ex $A = \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} \longrightarrow$ triangular matrix

$$\rightarrow \lambda = 3, 1$$

عناصر القطر الرئيسي.