3.8 Derivatives of Inverse Functions and Logarithms

In Section 1.6 we saw how the inverse of a function undoes, or inverts, the effect of that function. We defined there the natural logarithm function $f^{-1}(x) = \ln x$ as the inverse of the natural exponential function $f(x) = e^x$. This is one of the most important function-inverse pairs in mathematics and science. We learned how to differentiate the exponential function in Section 3.3. Here we learn a rule for differentiating the inverse of a differentiable function and we apply the rule to find the derivative of the natural logarithm function.

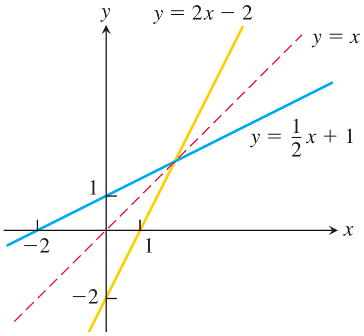


FIGURE 3.34 Graphing a line and its inverse together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other.

Derivatives of Inverses of Differentiable Functions

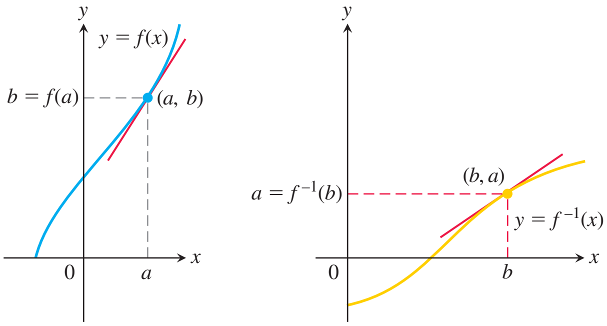
We calculated the inverse of the function $f(x) = (1/2)x + 1$ as $f^{-1}(x) = 2x - 2$ in Example 3 of Section 1.6. Figure 3.34 shows again the graphs of both functions. If we calculate their derivatives, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

The derivatives are reciprocals of one another, so the slope of one line is the reciprocal of the slope of its inverse line. (See Figure 3.34.)

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$, the reflected line has slope $1/m$.



The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

FIGURE 3.35 The graphs of inverse functions have reciprocal slopes at corresponding points.

The reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well, but we must be careful to compare slopes at corresponding points. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ and $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $1/f'(a)$ (Figure 3.35). If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$, then the inverse function f^{-1} has a vertical tangent line at $(f(a), a)$, and this infinite slope implies that f^{-1} is not differentiable at $f(a)$. Theorem 3 gives the conditions under which f^{-1} is differentiable in its domain (which is the same as the range of f).

THEOREM 3—The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Theorem 3 makes two assertions. The first of these has to do with the conditions under which f^{-1} is differentiable; the second assertion is a formula for the derivative of f^{-1} when it exists. While we omit the proof of the first assertion, the second one is proved in the following way:

$$f(f^{-1}(x)) = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}f(f^{-1}(x)) = 1 \qquad \text{Differentiating both sides}$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}f^{-1}(x) = 1 \qquad \text{Chain Rule}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}. \qquad \text{Solving for the derivative}$$

EXAMPLE 1 The function $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

Let's verify that Theorem 3 gives the same formula for the derivative of $f^{-1}(x)$:

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}\\ &= \frac{1}{2(f^{-1}(x))} \qquad f'(x) = 2x \text{ with } x \text{ replaced by } f^{-1}(x)\\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$

Theorem 3 gives a derivative that agrees with the known derivative of the square root function.

Let's examine Theorem 3 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 3 says that the derivative of f at 2, which is $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, which is $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \left.\frac{1}{2x}\right|_{x=2} = \frac{1}{4}.$$

See Figure 3.36. ■

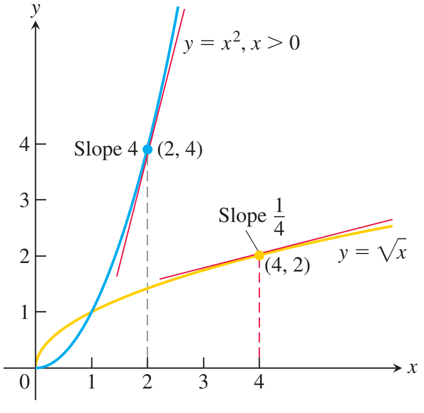


FIGURE 3.36 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 1).

We will use the procedure illustrated in Example 1 to calculate formulas for the derivatives of many inverse functions throughout this chapter. Equation (1) sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

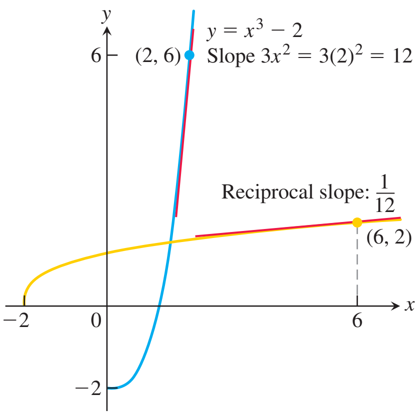


FIGURE 3.37 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 2).

EXAMPLE 2 Let $f(x) = x^3 - 2$, $x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution We apply Theorem 3 to obtain the value of the derivative of f^{-1} at $x = 6$:

$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12}. \qquad \text{Eq. (1)}$$

See Figure 3.37. ■

Derivative of the Natural Logarithm Function

Since we know the exponential function $f(x) = e^x$ is differentiable everywhere, we can apply Theorem 3 to find the derivative of its inverse $f^{-1}(x) = \ln x$:

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\ &= \frac{1}{e^{f^{-1}(x)}} && f'(u) = e^u \\ &= \frac{1}{e^{\ln x}} && x > 0 \\ &= \frac{1}{x}. && \text{Inverse function relationship} \end{aligned}$$

Alternate Derivation Instead of applying Theorem 3 directly, we can find the derivative of $y = \ln x$ using implicit differentiation, as follows:

$$\begin{aligned} y &= \ln x && x > 0 \\ e^y &= x && \text{Inverse function relationship} \\ \frac{d}{dx}(e^y) &= \frac{d}{dx}(x) && \text{Differentiate implicitly.} \\ e^y\frac{dy}{dx} &= 1 && \text{Chain Rule} \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}. && e^y = x \end{aligned}$$

No matter which derivation we use, the derivative of $y = \ln x$ with respect to x is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0.$$

The Chain Rule extends this formula to positive functions $u(x)$:

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \qquad (2)$$

EXAMPLE 3 We use Equation (2) to find derivatives.

(a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$

(b) Equation (2) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$

(c) Equation (2) with $u = |x|$ gives an important derivative:

$$\begin{aligned}\frac{d}{dx}\ln|x| &= \frac{d}{du}\ln u\cdot\frac{du}{dx} && u = |x|, x \neq 0\\ &= \frac{1}{u}\cdot\frac{x}{|x|} && \frac{d}{dx}(|x|) = \frac{x}{|x|}\\ &= \frac{1}{|x|}\cdot\frac{x}{|x|} && \text{Substitute for } u.\\ &= \frac{x}{x^2}\\ &= \frac{1}{x}.\end{aligned}$$

Derivative of ln |x|

$$\frac{d}{dx}\ln|x| = \frac{1}{x},\ x \neq 0$$

So $1/x$ is the derivative of $\ln x$ on the domain $x > 0$, and the derivative of $\ln(-x)$ on the domain $x < 0$. ■

Notice from Example 3a that the function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln bx$ for any constant b, provided that $bx > 0$:

$$\frac{d}{dx}\ln bx = \frac{1}{x}, \quad bx > 0$$

$$\frac{d}{dx}\ln bx = \frac{1}{bx}\cdot\frac{d}{dx}(bx) = \frac{1}{bx}(b) = \frac{1}{x}. \tag{3}$$

EXAMPLE 4 A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

Solution Suppose the point of tangency occurs at the unknown point $x = a > 0$. Then we know that the point $(a, \ln a)$ lies on the graph and that the tangent line at that point has slope $m = 1/a$ (Figure 3.38). Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

Setting these two formulas for m equal to each other, we have

$$\begin{aligned}\frac{\ln a}{a} &= \frac{1}{a}\\ \ln a &= 1\\ e^{\ln a} &= e^1\\ a &= e\\ m &= \frac{1}{e}.\end{aligned}$$ ■

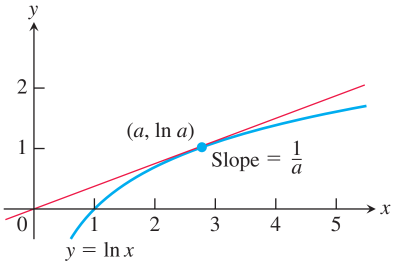


FIGURE 3.38 The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$ (Example 4).

The Derivatives of a^u and $\log_a u$

We start with the equation $a^x = e^{\ln(a^x)} = e^{x\ln a}$, $a > 0$, which was seen in Section 1.6:

$$\begin{aligned}\frac{d}{dx}a^x &= \frac{d}{dx}e^{x\ln a}\\ &= e^{x\ln a}\cdot\frac{d}{dx}(x\ln a) && \frac{d}{dx}e^u = e^u\frac{du}{dx}\\ &= a^x\ln a.\end{aligned}$$

That is, if $a > 0$, then a^x is differentiable and

$$\frac{d}{dx} a^x = a^x \ln a. \tag{4}$$

This equation shows why e^x is the preferred exponential function in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx} e^x = e^x \ln e = e^x.$$

With the Chain Rule, we get a more general form for the derivative of a general exponential function a^u.

> If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and
>
> $$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}. \tag{5}$$

EXAMPLE 5 Here are some derivatives of general exponential functions.

(a) $\frac{d}{dx} 3^x = 3^x \ln 3$ Eq. (5) with $a = 3, u = x$

(b) $\frac{d}{dx} 3^{-x} = 3^{-x}(\ln 3) \frac{d}{dx}(-x) = -3^{-x} \ln 3$ Eq. (5) with $a = 3, u = -x$

(c) $\frac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3) \frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$ $\ldots, u = \sin x$ ■

In Section 3.3 we looked at the derivative $f'(0)$ for the exponential functions $f(x) = a^x$ at various values of the base a. The number $f'(0)$ is the limit, $\lim_{h \to 0} (a^h - 1)/h$, and gives the slope of the graph of a^x when it crosses the y-axis at the point (0, 1). We now see from Equation (4) that the value of this slope is

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a. \tag{6}$$

In particular, when $a = e$ we obtain

$$\lim_{h \to 0} \frac{e^h - 1}{h} = \ln e = 1.$$

However, we have not fully justified that these limits actually exist. While all of the arguments given in deriving the derivatives of the exponential and logarithmic functions are correct, they do assume the existence of these limits. In Chapter 7 we will give another development of the theory of logarithmic and exponential functions which fully justifies that both limits do in fact exist and have the values derived above.

To find the derivative of $\log_a u$ for an arbitrary base ($a > 0, a \neq 1$), we start with the change-of-base formula for logarithms (reviewed in Section 1.6) and express $\log_a u$ in terms of natural logarithms,

$$\log_a x = \frac{\ln x}{\ln a}.$$

Taking derivatives, we have

$$\begin{aligned}\frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x \qquad \text{ln } a \text{ is a constant.}\\ &= \frac{1}{\ln a}\cdot\frac{1}{x}\\ &= \frac{1}{x\ln a}.\end{aligned}$$

If u is a differentiable function of x and $u > 0$, the Chain Rule gives a more general formula.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}. \tag{7}$$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

EXAMPLE 6 Find dy/dx if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the algebraic properties of logarithms from Theorem 1 in Section 1.6:

$$\begin{aligned}\ln y &= \ln\frac{(x^2+1)(x+3)^{1/2}}{x-1}\\ &= \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1) && \text{Rule 2}\\ &= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) && \text{Rule 1}\\ &= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). && \text{Rule 4}\end{aligned}$$

We then take derivatives of both sides with respect to x, using Equation (2) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Irrational Exponents and the Power Rule (General Version)

The definition of the general exponential function enables us to raise any positive number to any real power n, rational or irrational. That is, we can define the power function $y = x^n$ for any exponent n.

DEFINITION For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

Because the logarithm and exponential functions are inverses of each other, the definition gives

$$\ln x^n = n \ln x, \quad \text{for all real numbers } n.$$

That is, the rule for taking the natural logarithm of any power holds for *all* real exponents n, not just for rational exponents.

The definition of the power function also enables us to establish the derivative Power Rule for any real power n, as stated in Section 3.3.

General Power Rule for Derivatives

For $x > 0$ and any real number n,

$$\frac{d}{dx} x^n = nx^{n-1}.$$

If $x \le 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

Proof Differentiating x^n with respect to x gives

$$\begin{aligned}
\frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} && \text{Definition of } x^n, x > 0 \\
&= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) && \text{Chain Rule for } e^u \\
&= x^n \cdot \frac{n}{x} && \text{Definition and derivative of } \ln x \\
&= nx^{n-1}. && x^n \cdot x^{-1} = x^{n-1}
\end{aligned}$$

In short, whenever $x > 0$,

$$\frac{d}{dx} x^n = nx^{n-1}.$$

For $x < 0$, if $y = x^n$, y', and x^{n-1} all exist, then

$$\ln|y| = \ln|x|^n = n \ln|x|.$$

Using implicit differentiation (which *assumes* the existence of the derivative y') and Example 3(c), we have

$$\frac{y'}{y} = \frac{n}{x}.$$

Solving for the derivative,

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}. \qquad y = x^n$$

It can be shown directly from the definition of the derivative that the derivative equals 0 when $x = 0$ and $n \geq 1$ (see Exercise 99). This completes the proof of the general version of the Power Rule for all values of x. ■

EXAMPLE 7 Differentiate $f(x) = x^x$, $x > 0$.

Solution We note that $f(x) = x^x = e^{x\ln x}$, so differentiation gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x\ln x}) \\ &= e^{x\ln x}\frac{d}{dx}(x\ln x) && \tfrac{d}{dx}e^u,\ u = x\ln x \\ &= e^{x\ln x}\left(\ln x + x\cdot\frac{1}{x}\right) \\ &= x^x(\ln x + 1). && x > 0 \end{aligned}$$

We can also find the derivative of $y = x^x$ using logarithmic differentiation, assuming y' exists. ■

The Number e Expressed as a Limit

In Section 1.5 we defined the number e as the base value for which the exponential function $y = a^x$ has slope 1 when it crosses the y-axis at (0, 1). Thus e is the constant that satisfies the equation

$$\lim_{h\to 0}\frac{e^h - 1}{h} = \ln e = 1. \qquad \text{Slope equals } \ln e \text{ from Eq. (6).}$$

We now prove that e can be calculated as a certain limit.

THEOREM 4—The Number e as a Limit The number e can be calculated as the limit

$$e = \lim_{x\to 0}(1 + x)^{1/x}.$$

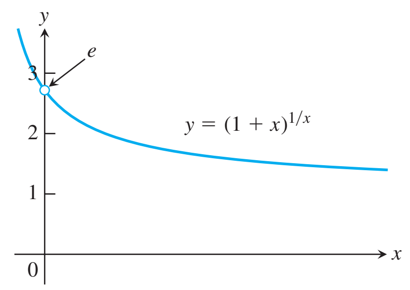


FIGURE 3.39 The number e is the limit of the function graphed here as $x \to 0$.

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x) && \ln 1 = 0 \\ &= \lim_{x\to 0}\ln(1+x)^{1/x} = \ln\left[\lim_{x\to 0}(1+x)^{1/x}\right]. && \ln \text{ is continuous, Theorem 10 in Chapter 2.} \end{aligned}$$

Because $f'(1) = 1$, we have

$$\ln\left[\lim_{x\to 0}(1 + x)^{1/x}\right] = 1.$$

Therefore, exponentiating both sides we get

$$\lim_{x\to 0}(1 + x)^{1/x} = e.$$

See Figure 3.39 on the previous page. ■

Approximating the limit in Theorem 4 by taking x very small gives approximations to e. Its value is $e \approx 2.718281828459045$ to 15 decimal places.

Exercises 3.8

Derivatives of Inverse Functions

In Exercises 1–4:

a. Find $f^{-1}(x)$.

b. Graph f and f^{-1} together.

c. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

1. $f(x) = 2x + 3, \quad a = -1$ **2.** $f(x) = (1/5)x + 7, \quad a = -1$

3. $f(x) = 5 - 4x, \quad a = 1/2$ **4.** $f(x) = 2x^2, \quad x \ge 0, \quad a = 5$

5. a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

b. Graph f and g over an x-interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs of f and g at $(1, 1)$ and $(-1, -1)$ (four tangents in all).

d. What lines are tangent to the curves at the origin?

6. a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

b. Graph h and k over an x-interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs at h and k at $(2, 2)$ and $(-2, -2)$.

d. What lines are tangent to the curves at the origin?

7. Let $f(x) = x^3 - 3x^2 - 1, x \ge 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

8. Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

9. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at $x = 4$.

10. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

Derivatives of Logarithms

In Exercises 11–40, find the derivative of y with respect to x, t, or θ, as appropriate.

11. $y = \ln 3x + x$ **12.** $y = \dfrac{1}{\ln 3x}$

13. $y = \ln(t^2)$ **14.** $y = \ln(t^{3/2}) + \sqrt{t}$

15. $y = \ln\dfrac{3}{x}$ **16.** $y = \ln(\sin x)$

17. $y = \ln(\theta + 1) - e^{\theta}$ **18.** $y = (\cos\theta)\ln(2\theta + 2)$

19. $y = \ln x^3$ **20.** $y = (\ln x)^3$

21. $y = t(\ln t)^2$ **22.** $y = t\ln\sqrt{t}$

23. $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$ **24.** $y = (x^2\ln x)^4$

25. $y = \dfrac{\ln t}{t}$ **26.** $y = \dfrac{t}{\sqrt{\ln t}}$

27. $y = \dfrac{\ln x}{1 + \ln x}$ **28.** $y = \dfrac{x\ln x}{1 + \ln x}$

29. $y = \ln(\ln x)$ **30.** $y = \ln(\ln(\ln x))$

31. $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$

32. $y = \ln(\sec\theta + \tan\theta)$

33. $y = \ln\dfrac{1}{x\sqrt{x + 1}}$ **34.** $y = \dfrac{1}{2}\ln\dfrac{1 + x}{1 - x}$

35. $y = \dfrac{1 + \ln t}{1 - \ln t}$ **36.** $y = \sqrt{\ln\sqrt{t}}$

37. $y = \ln(\sec(\ln\theta))$ **38.** $y = \ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$

39. $y = \ln\left(\dfrac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$ **40.** $y = \ln\sqrt{\dfrac{(x + 1)^5}{(x + 2)^{20}}}$

Logarithmic Differentiation

In Exercises 41–54, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

41. $y = \sqrt{x(x + 1)}$ **42.** $y = \sqrt{(x^2 + 1)(x - 1)^2}$

43. $y = \sqrt{\dfrac{t}{t + 1}}$ **44.** $y = \sqrt{\dfrac{1}{t(t + 1)}}$

45. $y = (\sin\theta)\sqrt{\theta + 3}$ **46.** $y = (\tan\theta)\sqrt{2\theta + 1}$

47. $y = t(t + 1)(t + 2)$ **48.** $y = \dfrac{1}{t(t + 1)(t + 2)}$

49. $y = \dfrac{\theta + 5}{\theta\cos\theta}$ **50.** $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

51. $y = \dfrac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}$ **52.** $y = \sqrt{\dfrac{(x + 1)^{10}}{(2x + 1)^5}}$

53. $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$ **54.** $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Finding Derivatives

In Exercises 55–62, find the derivative of y with respect to x, t, or θ, as appropriate.

55. $y = \ln(\cos^2\theta)$ **56.** $y = \ln(3\theta e^{-\theta})$

57. $y = \ln(3te^{-t})$ **58.** $y = \ln(2e^{-t}\sin t)$

59. $y = \ln\left(\dfrac{e^\theta}{1+e^\theta}\right)$ **60.** $y = \ln\left(\dfrac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)$

61. $y = e^{(\cos t + \ln t)}$ **62.** $y = e^{\sin t}(\ln t^2 + 1)$

In Exercises 63–66, find dy/dx.

63. $\ln y = e^y \sin x$ **64.** $\ln xy = e^{x+y}$

65. $x^y = y^x$ **66.** $\tan y = e^x + \ln x$

In Exercises 67–88, find the derivative of y with respect to the given independent variable.

67. $y = 2^x$ **68.** $y = 3^{-x}$

69. $y = 5^{\sqrt{s}}$ **70.** $y = 2^{(s^2)}$

71. $y = x^\pi$ **72.** $y = t^{1-e}$

73. $y = \log_2 5\theta$ **74.** $y = \log_3(1 + \theta\ln 3)$

75. $y = \log_4 x + \log_4 x^2$ **76.** $y = \log_{25} e^x - \log_5\sqrt{x}$

77. $y = \log_2 r \cdot \log_4 r$ **78.** $y = \log_3 r \cdot \log_9 r$

79. $y = \log_3\left(\left(\dfrac{x+1}{x-1}\right)^{\ln 3}\right)$ **80.** $y = \log_5\sqrt{\left(\dfrac{7x}{3x+2}\right)^{\ln 5}}$

81. $y = \theta\sin(\log_7\theta)$ **82.** $y = \log_7\left(\dfrac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right)$

83. $y = \log_5 e^x$ **84.** $y = \log_2\left(\dfrac{x^2e^2}{2\sqrt{x+1}}\right)$

85. $y = 3^{\log_2 t}$ **86.** $y = 3\log_8(\log_2 t)$

87. $y = \log_2(8t^{\ln 2})$ **88.** $y = t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

Logarithmic Differentiation with Exponentials

In Exercises 89–96, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

89. $y = (x+1)^x$ **90.** $y = x^{(x+1)}$

91. $y = (\sqrt{t})^t$ **92.** $y = t^{\sqrt{t}}$

93. $y = (\sin x)^x$ **94.** $y = x^{\sin x}$

95. $y = x^{\ln x}$ **96.** $y = (\ln x)^{\ln x}$

Theory and Applications

97. If we write $g(x)$ for $f^{-1}(x)$, Equation (1) can be written as

$$g'(f(a)) = \frac{1}{f'(a)}, \quad \text{or} \quad g'(f(a)) \cdot f'(a) = 1.$$

If we then write x for a, we get

$$g'(f(x)) \cdot f'(x) = 1.$$

The latter equation may remind you of the Chain Rule, and indeed there is a connection.

Assume that f and g are differentiable functions that are inverses of one another, so that $(g \circ f)(x) = x$. Differentiate both sides of this equation with respect to x, using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of g and f. What do you find? (This is not a proof of Theorem 3 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

98. Show that $\lim_{n\to\infty}\left(1 + \dfrac{x}{n}\right)^n = e^x$ for any $x > 0$.

99. If $f(x) = x^n$, $n \geq 1$, show from the definition of the derivative that $f'(0) = 0$.

100. Using mathematical induction, show that for $n > 1$

$$\frac{d^n}{dx^n}\ln x = (-1)^{n-1}\frac{(n-1)!}{x^n}.$$

COMPUTER EXPLORATIONS

In Exercises 101–108, you will explore some functions and their inverses together with their derivatives and tangent line approximations at specified points. Perform the following steps using your CAS:

a. Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that f is one-to-one over the interval.

b. Solve the equation $y = f(x)$ for x as a function of y, and name the resulting inverse function g.

c. Find the equation for the tangent line to f at the specified point $(x_0, f(x_0))$.

d. Find the equation for the tangent line to g at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 3 to find the slope of this tangent line.

e. Plot the functions f and g, the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal.

101. $y = \sqrt{3x-2}, \quad \dfrac{2}{3} \leq x \leq 4, \quad x_0 = 3$

102. $y = \dfrac{3x+2}{2x-11}, \quad -2 \leq x \leq 2, \quad x_0 = 1/2$

103. $y = \dfrac{4x}{x^2+1}, \quad -1 \leq x \leq 1, \quad x_0 = 1/2$

104. $y = \dfrac{x^3}{x^2+1}, \quad -1 \leq x \leq 1, \quad x_0 = 1/2$

105. $y = x^3 - 3x^2 - 1, \quad 2 \leq x \leq 5, \quad x_0 = \dfrac{27}{10}$

106. $y = 2 - x - x^3, \quad -2 \leq x \leq 2, \quad x_0 = \dfrac{3}{2}$

107. $y = e^x, \quad -3 \leq x \leq 5, \quad x_0 = 1$

108. $y = \sin x, \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}, \quad x_0 = 1$

In Exercises 109 and 110, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

109. $y^{1/3} - 1 = (x+2)^3, \quad -5 \leq x \leq 5, \quad x_0 = -3/2$

110. $\cos y = x^{1/5}, \quad 0 \leq x \leq 1, \quad x_0 = 1/2$

3.9 Inverse Trigonometric Functions

We introduced the six basic inverse trigonometric functions in Section 1.6, but focused there on the arcsine and arccosine functions. Here we complete the study of how all six inverse trigonometric functions are defined, graphed, and evaluated, and how their derivatives are computed.

Inverses of tan *x*, cot *x*, sec *x*, and csc *x*

The graphs of these four basic inverse trigonometric functions are shown again in Figure 3.40. We obtain these graphs by reflecting the graphs of the restricted trigonometric functions (as discussed in Section 1.6) through the line $y = x$. Let's take a closer look at the arctangent, arccotangent, arcsecant, and arccosecant functions.

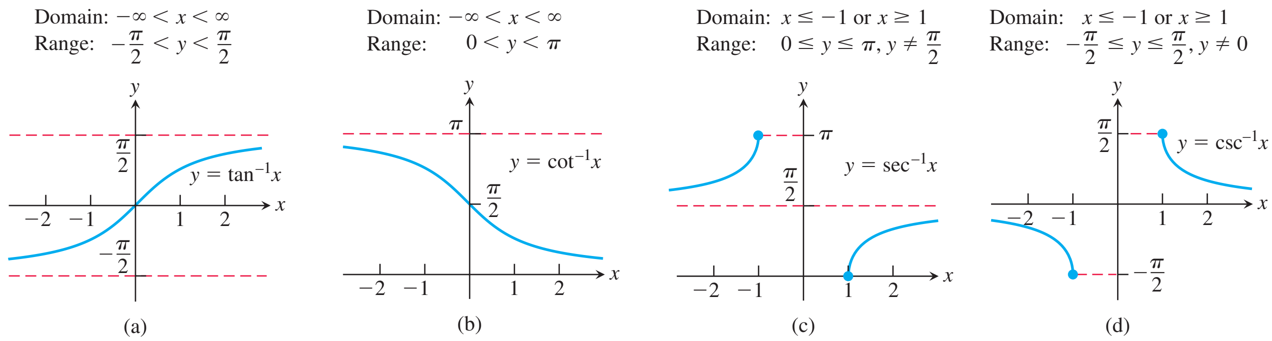


FIGURE 3.40 Graphs of the arctangent, arccotangent, arcsecant, and arccosecant functions.

The arctangent of x is a radian angle whose tangent is x. The arccotangent of x is an angle whose cotangent is x, and so forth. The angles belong to the restricted domains of the tangent, cotangent, secant, and cosecant functions.

> **DEFINITIONS**
>
> $y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.
>
> $y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.
>
> $y = \sec^{-1} x$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.
>
> $y = \csc^{-1} x$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

We use open or half-open intervals to avoid values for which the tangent, cotangent, secant, and cosecant functions are undefined. (See Figure 3.40.)

The graph of $y = \tan^{-1} x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Figure 3.40a). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1} x;$$

the arctangent is an odd function. The graph of $y = \cot^{-1} x$ has no such symmetry (Figure 3.40b). Notice from Figure 3.40a that the graph of the arctangent function has two horizontal asymptotes: one at $y = \pi/2$ and the other at $y = -\pi/2$.

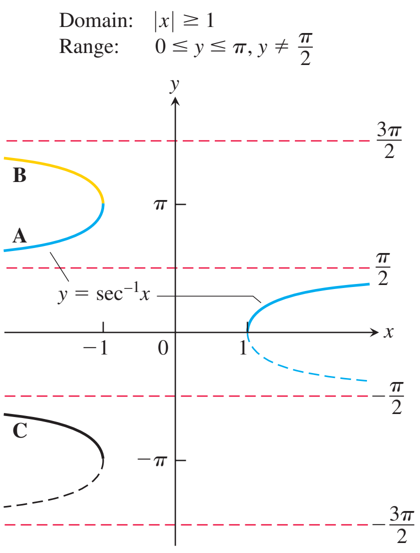


FIGURE 3.41 There are several logical choices for the left-hand branch of $y = \sec^{-1}x$. With choice **A**, $\sec^{-1}x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

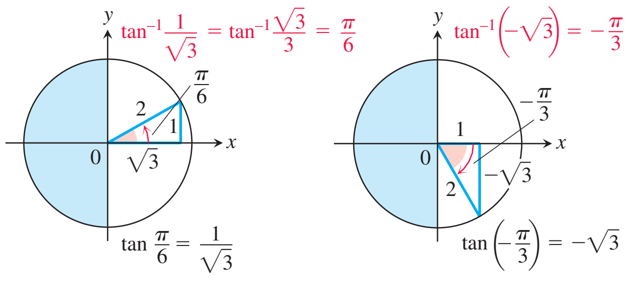
The inverses of the restricted forms of sec x and csc x are chosen to be the functions graphed in Figures 3.40c and 3.40d.

Caution There is no general agreement about how to define $\sec^{-1}x$ for negative values of x. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $\sec^{-1}x = \cos^{-1}(1/x)$. It also makes $\sec^{-1}x$ an increasing function on each interval of its domain. Some tables choose $\sec^{-1}x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 3.41). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1}x = \cos^{-1}(1/x)$. From this, we can derive the identity

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right) \qquad (1)$$

by applying Equation (5) in Section 1.6.

EXAMPLE 1 The accompanying figures show two values of $\tan^{-1}x$.



x	$\tan^{-1}x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

The angles come from the first and fourth quadrants because the range of $\tan^{-1}x$ is $(-\pi/2, \pi/2)$. ■

The Derivative of $y = \sin^{-1}u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 3 in Section 3.8 therefore assures us that the inverse function $y = \sin^{-1}x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points (see Figure 3.42).

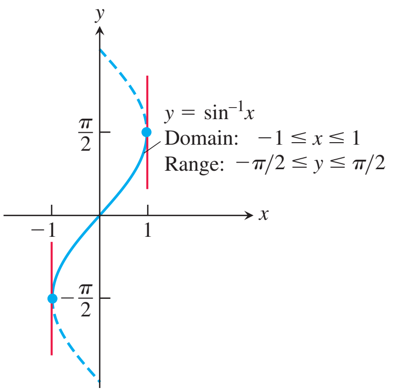


FIGURE 3.42 The graph of $y = \sin^{-1}x$ has vertical tangents at $x = -1$ and $x = 1$.

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 3 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$:

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\
&= \frac{1}{\sqrt{1 - x^2}}. && \sin(\sin^{-1} x) = x
\end{aligned}$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get the general formula

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \qquad |u| < 1.$$

EXAMPLE 2 Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}.$$

The Derivative of $y = \tan^{-1} u$

We find the derivative of $y = \tan^{-1} x$ by applying Theorem 3 with $f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$. Theorem 3 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$:

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u \\
&= \frac{1}{1 + \tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u \\
&= \frac{1}{1 + x^2}. && \tan(\tan^{-1} x) = x
\end{aligned}$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

The Derivative of $y = \sec^{-1} u$

Since the derivative of $\sec x$ is positive for $0 < x < \pi/2$ and $\pi/2 < x < \pi$, Theorem 3 says that the inverse function $y = \sec^{-1} x$ is differentiable. Instead of applying the formula in Theorem 3 directly, we find the derivative of $y = \sec^{-1} x$, $|x| > 1$, using implicit differentiation and the Chain Rule as follows:

$$\begin{aligned} y &= \sec^{-1} x \\ \sec y &= x && \text{Inverse function relationship} \\ \frac{d}{dx}(\sec y) &= \frac{d}{dx} x && \text{Differentiate both sides.} \\ \sec y \tan y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y}. && \text{Since } |x| > 1, y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0. \end{aligned}$$

To express the result in terms of x, we use the relationships

$$\sec y = x \qquad \text{and} \qquad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2 - 1}}.$$

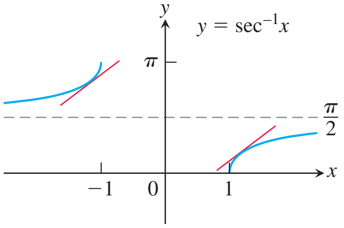


FIGURE 3.43 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

Can we do anything about the $\pm$ sign? A glance at Figure 3.43 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. Thus,

$$\frac{d}{dx}\sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the "$\pm$" ambiguity:

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

EXAMPLE 3 Using the Chain Rule and derivative of the arcsecant function, we find

$$\begin{aligned} \frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4) \\ &= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) && 5x^4 > 1 > 0 \\ &= \frac{4}{x\sqrt{25x^8 - 1}}. \end{aligned}$$

Derivatives of the Other Three Inverse Trigonometric Functions

We could use the same techniques to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is an easier way, thanks to the following identities.

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

We saw the first of these identities in Equation (5) of Section 1.6. The others are derived in a similar way. It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, the derivative of $\cos^{-1} x$ is calculated as follows:

$$\frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) \qquad \text{Identity}$$
$$= -\frac{d}{dx}(\sin^{-1} x)$$
$$= -\frac{1}{\sqrt{1 - x^2}}. \qquad \text{Derivative of arcsine}$$

The derivatives of the inverse trigonometric functions are summarized in Table 3.1.

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1 - u^2}}\dfrac{du}{dx}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1 - u^2}}\dfrac{du}{dx}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1 + u^2}\dfrac{du}{dx}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1 + u^2}\dfrac{du}{dx}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{|u|\sqrt{u^2 - 1}}\dfrac{du}{dx}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2 - 1}}\dfrac{du}{dx}, \quad |u| > 1$

Exercises 3.9

Common Values

Use reference triangles in an appropriate quadrant, as in Example 1, to find the angles in Exercises 1–8.

1. a. $\tan^{-1} 1$ **b.** $\tan^{-1}(-\sqrt{3})$ **c.** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

2. a. $\tan^{-1}(-1)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

3. a. $\sin^{-1}\left(\frac{-1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

4. a. $\sin^{-1}\left(\frac{1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

5. a. $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

6. a. $\csc^{-1}\sqrt{2}$ **b.** $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ **c.** $\csc^{-1} 2$

7. a. $\sec^{-1}(-\sqrt{2})$ **b.** $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **c.** $\sec^{-1}(-2)$

8. a. $\cot^{-1}(-1)$ **b.** $\cot^{-1}(\sqrt{3})$ **c.** $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Evaluations

Find the values in Exercises 9–12.

9. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

10. $\sec\left(\cos^{-1}\frac{1}{2}\right)$

11. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

12. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Limits

Find the limits in Exercises 13–20. (If in doubt, look at the function's graph.)

13. $\lim_{x\to 1^-} \sin^{-1} x$

14. $\lim_{x\to -1^+} \cos^{-1} x$

15. $\lim_{x\to\infty} \tan^{-1} x$

16. $\lim_{x\to -\infty} \tan^{-1} x$

17. $\lim_{x\to\infty} \sec^{-1} x$

18. $\lim_{x\to -\infty} \sec^{-1} x$

19. $\lim_{x\to\infty} \csc^{-1} x$

20. $\lim_{x\to -\infty} \csc^{-1} x$

Finding Derivatives

In Exercises 21–42, find the derivative of y with respect to the appropriate variable.

21. $y = \cos^{-1}(x^2)$

22. $y = \cos^{-1}(1/x)$

23. $y = \sin^{-1}\sqrt{2}\,t$

24. $y = \sin^{-1}(1 - t)$

25. $y = \sec^{-1}(2s + 1)$

26. $y = \sec^{-1} 5s$

27. $y = \csc^{-1}(x^2 + 1), \quad x > 0$

28. $y = \csc^{-1}\frac{x}{2}$

29. $y = \sec^{-1}\frac{1}{t}, \quad 0 < t < 1$

30. $y = \sin^{-1}\frac{3}{t^2}$

31. $y = \cot^{-1}\sqrt{t}$

32. $y = \cot^{-1}\sqrt{t - 1}$

33. $y = \ln(\tan^{-1} x)$

34. $y = \tan^{-1}(\ln x)$

35. $y = \csc^{-1}(e^t)$

36. $y = \cos^{-1}(e^{-t})$

37. $y = s\sqrt{1 - s^2} + \cos^{-1} s$

38. $y = \sqrt{s^2 - 1} - \sec^{-1} s$

39. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x, \quad x > 1$

40. $y = \cot^{-1}\frac{1}{x} - \tan^{-1} x$

41. $y = x\sin^{-1} x + \sqrt{1 - x^2}$

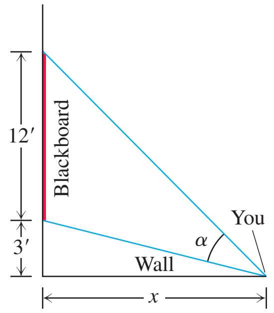
42. $y = \ln(x^2 + 4) - x\tan^{-1}\left(\frac{x}{2}\right)$

Theory and Examples

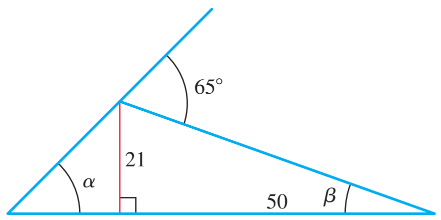
43. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

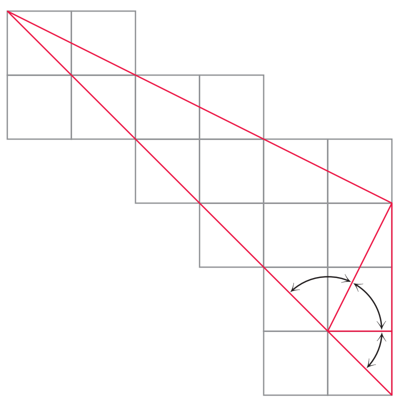
if you are x ft from the front wall.



44. Find the angle α.

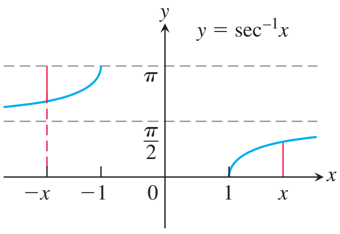


45. Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



46. Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$

a. (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1} x$. See if you can tell what is going on.



b. (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad \text{Eq. (4), Section 1.6}$$

$$\sec^{-1} x = \cos^{-1}(1/x) \qquad \text{Eq. (1)}$$

Which of the expressions in Exercises 47–50 are defined, and which are not? Give reasons for your answers.

47. a. $\tan^{-1} 2$ **b.** $\cos^{-1} 2$

48. a. $\csc^{-1}(1/2)$ **b.** $\csc^{-1} 2$

49. a. $\sec^{-1} 0$ **b.** $\sin^{-1}\sqrt{2}$

50. a. $\cot^{-1}(-1/2)$ **b.** $\cos^{-1}(-5)$

51. Use the identity

$$\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u$$

to derive the formula for the derivative of $\csc^{-1} u$ in Table 3.1 from the formula for the derivative of $\sec^{-1} u$.

52. Derive the formula

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

for the derivative of $y = \tan^{-1} x$ by differentiating both sides of the equivalent equation $\tan y = x$.

53. Use the Derivative Rule in Section 3.8, Theorem 3, to derive

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}, \qquad |x| > 1.$$

54. Use the identity

$$\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u$$

to derive the formula for the derivative of $\cot^{-1} u$ in Table 3.1 from the formula for the derivative of $\tan^{-1} u$.

55. What is special about the functions

$$f(x) = \sin^{-1}\frac{x - 1}{x + 1}, \quad x \geq 0, \quad \text{and} \quad g(x) = 2\tan^{-1}\sqrt{x}?$$

Explain.

56. What is special about the functions

$$f(x) = \sin^{-1}\frac{1}{\sqrt{x^2 + 1}} \quad \text{and} \quad g(x) = \tan^{-1}\frac{1}{x}?$$

Explain.

T **57.** Find the values of

a. $\sec^{-1} 1.5$ **b.** $\csc^{-1}(-1.5)$ **c.** $\cot^{-1} 2$

T **58.** Find the values of

a. $\sec^{-1}(-3)$ **b.** $\csc^{-1} 1.7$ **c.** $\cot^{-1}(-2)$

T In Exercises 59–61, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

59. a. $y = \tan^{-1}(\tan x)$ **b.** $y = \tan(\tan^{-1} x)$

60. a. $y = \sin^{-1}(\sin x)$ **b.** $y = \sin(\sin^{-1} x)$

61. a. $y = \cos^{-1}(\cos x)$ **b.** $y = \cos(\cos^{-1} x)$

T Use your graphing utility for Exercises 62–66.

62. Graph $y = \sec(\sec^{-1} x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

63. Newton's serpentine Graph Newton's serpentine, $y = 4x/(x^2 + 1)$. Then graph $y = 2\sin(2\tan^{-1} x)$ in the same graphing window. What do you see? Explain.

64. Graph the rational function $y = (2 - x^2)/x^2$. Then graph $y = \cos(2\sec^{-1} x)$ in the same graphing window. What do you see? Explain.

65. Graph $f(x) = \sin^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

66. Graph $f(x) = \tan^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

3.10 Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a *related rates problem*.