complete sec. 3.2

The span of a set of vectors

Def: V is a vector space:

let $v_1, v_2, \ldots v_n \in V$,

the sum of the form:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

scalar vectors

where $\alpha_1, \alpha_2, \ldots \alpha_n$ scalars.

is called linear combination of the vectors $v_1, v_2, \ldots v_n$

The set of all linear combinations of $v_1, v_2, \ldots, v_n$ is called the span of $v_1, v_2, \ldots, v_n$.

Standard vectors

In R^2: $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$

In R^3: $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\vdots$

In R^n: $e_1, e_2, \ldots e_n$ $\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}$

ex $R^{2\times 2}$: $E_{11} = \begin{bmatrix}1&0\\0&0\end{bmatrix}$, $E_{12} = \begin{bmatrix}0&1\\0&0\end{bmatrix}$, $E_{21} = \begin{bmatrix}0&0\\1&0\end{bmatrix}$, $E_{22} = \begin{bmatrix}0&0\\0&1\end{bmatrix}$

ex In $\mathbb{R}^3$: find,

$\text{span}(e_1, e_2) = ?$

Sol: $\text{span}(e_1, e_2) = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

$= \alpha_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \alpha_1, \alpha_2 \in \mathbb{R}.$

$= \begin{bmatrix}\alpha_1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\\alpha_2\\0\end{bmatrix} = \left\{\begin{bmatrix}\alpha_1\\\alpha_2\\0\end{bmatrix}, \quad \alpha_1, \alpha_2 \in \mathbb{R}\right\}$

ex In $\mathbb{R}^3$, find.

$\text{span}(e_1, e_2, e_3) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$= \alpha_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + \alpha_3 \begin{bmatrix}0\\0\\1\end{bmatrix} =$

$= \left\{ \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\} = \mathbb{R}^3$

Note:

In $\mathbb{R}^2$, $\text{span}(\underset{\begin{bmatrix}1\\0\end{bmatrix}}{e_1}, \underset{\begin{bmatrix}0\\1\end{bmatrix}}{e_2}) = \mathbb{R}^2$

In $\mathbb{R}^3$, $\text{span}(e_1, e_2, e_3) = \mathbb{R}^3$

In $\mathbb{R}^4$, $\text{span}(e_1, \underset{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}{e_2}, e_3, e_4) = \mathbb{R}^4$

$\vdots$

In $\mathbb{R}^n$, $\text{span}(e_1, e_2, \ldots, e_n) = \mathbb{R}^n$

Theorem:

let (V) be a vector space,

$v_1, v_2, \dots v_n \in V$, then

$\operatorname{span}(v_1, v_2, \dots v_n)$ is a subspace of V

Remark

let V be a vector space, then

$\{0\}, V$ are subspaces of V

Spanning Set of a vector Space

Def:

The set $\{v_1, v_2, \dots v_n\}$ is a spanning set for V $\left(\operatorname{span}(v_1, v_2, \dots v_n) = V\right)$

if and only if every vector in V can be written as a linear combination of $v_1, v_2, \dots, v_n$.

ex Which of the following are spanning set for R^3?

(a) $\{e_1, e_2, e_3\}$

Sol: any vector in $V = R^3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix} = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

$$= 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + -1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 8 \\ 2 \end{bmatrix} = 10e_1 + 8e_2 + 2e_3.$$

$$\begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} = 0e_1 + 0e_2 + 5e_3$$

كل vector نكتبه من تركيب خطي من عناصر المجموعة

$\rightarrow$ the set $\{e_1, e_2, e_3\}$ is a spanning set for R^3.

Note: In R^2, $\{e_1, e_2\}$ spanning set for R^2

In R^3, $\{e_1, e_2, e_3\}$ ~ ~ ~ R^3.

⋮

In R^n, $\{e_1, e_2, \dots e_n\}$ ~ ~ ~ R^n.

(b) $\{e_1, e_2, e_3, (1,2,3)^T\}$
spanning set ~~for~~ for $\mathbb{R}^3$??

Sol: any vector in $\mathbb{R}^3 \to \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \overset{??}{=} \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Find $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

the sys. has a sol. $\downarrow$ spanning set for V

the system inconsistent (don't have a sol) $\downarrow$ not spanning set for V

$\alpha_1 = a, \; \alpha_2 = b, \; \alpha_3 = c, \; \alpha_4 = 0$

$\to \{e_1, e_2, e_3, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ spanning set for $\mathbb{R}^3$.

(c) $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$.

Sol: any vector in $\mathbb{R}^3 \to \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \overset{??}{=} \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Find $\alpha_1, \alpha_2, \alpha_3$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 1 & 0 & b \\ 1 & 0 & 0 & c \end{array}\right] \longrightarrow$$

sol $\to$ spanning set

no sol. $\to$ not spanning

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 1 & 0 & b \\ 1 & 0 & 0 & c \end{array}\right] \Rightarrow \boxed{\alpha_1 = c}$$

$\alpha_1 + \alpha_2 = b$

$c + \alpha_2 = b$

$\rightarrow \boxed{\alpha_2 = b - c}$

$\alpha_1 + \alpha_2 + \alpha_3 = a$

$c + b - c + \alpha_3 = a \Rightarrow \boxed{\alpha_3 = a - b}$

$\rightarrow$ this system has a sol. α_1 ✓ α_2 ✓ α_3 ✓

$\rightarrow$ the set $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$ is a spanning set for R^3.

(d) $\{(1,0,1)^T, (0,1,0)^T\}$.

معلومة من المحاضرات القادمة : اي مجموعة عدد عناصرها أقل من (3) في R^3 ليست spanning set for R^3.

any vector in $R^3 \longrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \overset{??}{=} \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$ find α_1, α_2.

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ 1 & 0 & c \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c-a \end{array}\right],$$

in general $c - a \neq 0 \Rightarrow [0 \quad 0 \mid \underbrace{\text{number}}_{\neq 0}] \Rightarrow$

the system is inconsistent $\Rightarrow$ not spannin set for R^3.

(e) $\{ (1,2,4)^T, (2,1,3)^T, (4,-1,1)^T \}$

sol. any vector in $R^3 \to \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \overset{??}{=} \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

find $\alpha_1, \alpha_2, \alpha_3$ which is the sol.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 2 & 1 & -1 & b \\ 4 & 3 & 1 & c \end{array}\right] \xrightarrow{REF}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & 1 & 3 & \frac{2a-b}{3} \\ 0 & 0 & 0 & 2a-3c+5b \end{array}\right]$$

in general $2a - 3c + 5b \neq 0$

$\to$ this system is inconsistent

$\to$ the set $\{ (1,2,4)^T, \text{---} \}$ is **not** a spanning set for R^3.

Q11 spanning set for R^2.

$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

sol $\begin{bmatrix} a \\ b \end{bmatrix} \to \begin{bmatrix} a \\ b \end{bmatrix} \overset{??}{=} \alpha_1 [\;] + \alpha_2 [\;]$

$$\left[\begin{array}{cc|c} 2 & 3 & a \\ 1 & 2 & b \end{array}\right]$$

$\to$ sol. $\to$ span

$\to$ no sol. $\to$ not

ex $\{1-x^2, x+2, x^2\}$

spanning set for P_3 ??

sol: P_3 : polynomial at order less than 3

$\rightarrow ax^2 + bx + c.$

$$ax^2+bx+C \overset{??}{=} \alpha_1(1-x^2)+\alpha_2(x+2) + \alpha_3(x^2).$$

$$ax^2+bx+C = (\alpha_3-\alpha_1)x^2 + \alpha_2 x + (\alpha_1+2\alpha_2)$$

$$\alpha_3 - \alpha_1 = a$$
$$\alpha_2 = b$$
$$\alpha_1 + 2\alpha_2 = C.$$

find $\alpha_1, \alpha_2, \alpha_3, \ldots$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & a \\ 0 & 1 & 0 & b \\ 1 & 2 & 0 & c \end{array}\right] \longrightarrow \text{REF}$$

$$\alpha_1 = c - 2b$$
$$\alpha_2 = b$$
$$\alpha_3 = a + c - 2b.$$

$\rightarrow$ spanning set for P_3.

Q13 Given

$x_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \quad x = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}, \quad y = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix}$

(a) Is $x \in \text{span}(x_1, x_2)$?

sol:

$$\begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix} \overset{?}{\in} \text{span}\left(\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \right).$$

$$\begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix} \overset{?}{\in} \alpha_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

find the sol. of the system

sol. → $\in$ span

no sol. → $\notin$ span.

$$\left[\begin{array}{cc|c} -1 & 3 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 10 & 10 \\ 0 & 11 & 12 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 11 & 12 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

→ this sys. is inconsistent.

→ $x \notin \text{span}(x_1, x_2)$.

(b) Is $y \in \text{span}(x_1, x_2)$.

$y \in \text{span}$ ← لحل ← واجب

Q14

$\{x_1, x_2, \_\_\_ x_k\}$ spanning set for V

add (x_{k+1}) $\longrightarrow$ still spanning set V.

delete( ) $\longrightarrow$?? $\nearrow$ span

$\searrow$ not span.