Lecture Notes on Numerical Differential Equations: IVP

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1 Initial Value Problem for Ordinary Differential Equations

We consider the problem of numerically solving a system of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha \text{ (given) } .$$

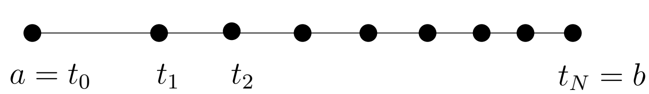
Such a problem is called the **Initial Value Problem** or in short **IVP**, because the initial value of the solution $y(a) = \alpha$ is given.

Since there are infinitely many values between a and b, we will only be concerned here to find approximations of the solution $y(t)$ at several specified values of t in $[a, b]$, rather than finding $y(t)$ at every value between a and b.

Denote

- $y_i =$ (an approximate value of $y(at)$ at $t = t_i$.)
- Divide $[a, b]$ into N equal subintervals of length h:

$$t_0 = a < t_1 < t_2 < \cdots t_N = b.$$



- $h = \dfrac{b-a}{N}$ **(step size)**

The Initial Value Problem
Given

(1) $y' = f(y, t),\ a \le t \le b$

(2) The initial value $y(t_0) = y(a) = \alpha$

(3) The step-size h.

Find y_i (an approximation of $y(t_i)$), $i = 1, \cdots, N$, where $N = \dfrac{b-a}{h}$.

We will briefly describe here the well-known numerical methods for solving the IVP, such as the

- The **Euler Method**
- The **Taylor Method** of higher order
- The **Runge-Kutta Method**
- The **Adams-Moulton Method**
- The **Milne Method**

etc.

We will also discuss the error behavior and convergence of these methods.

However, before doing so, we state a result **without proof**, in the following section on the **existence** and **uniqueness** of the solution for the IVP. The proof can be found in most books on ordinary differential equations.

Existence and Uniqueness of the Solution for the IVP

Theorem: (Existence and Uniqueness Theorem for the IVP).

The initial value problem:

$$\begin{cases} y' = f(t,y) \\ y(a) = \alpha \end{cases}$$

has a unique solution $y(t)$ for $a \le t \le b$, if $f(t,y)$ is continuous on the domain, given by $R = \{a \le t \le b, \quad \infty < y < \infty\}$ and satisfies the following inequality:

$$|f(t,y) - f(t,y^*)| \le L|y - y^*|,$$

Whenever (t,y) and $(t,y^*) \in R$.

■

Definition. The condition $|f(t,y) - f(t,y^*)| \le L|y - y^*|$ is called the **Lipschitz Condition**. The number L is called a **Lipschitz Constant**.

Definition.

A set S is said to be convex if whenever (t_1, y_1) and (t_2, y_2) belong to S, the point $\Big((1-\lambda)t_1 + 2t_2,\ (1-\lambda)y_1 + \lambda y_2)$ also belongs to S for each λ when $0 \le \lambda \le 1$.

Simplification of the Lipschitz Condition for the Convex Domain

If the domain happens to be a **convex set**, then the condition of the above Theorem reduces to

$$\left|\frac{\partial f}{\partial y}(t,y)\right| \le L \text{ for all } (t,y) \in R.$$

Liptischitz Condition and Well-Posednes

Definition.

An IVP is said to be **well-posed** if a small perhubation in the data of the problem leads to only a small change in the solution.

Since numerical computation may very well introduce some perhubations to the problem, it is important that the problem that is to be solved is well-posed.

Fortunately, the Lipschitz condition is a sufficient condition for the IVP problem to be well-posed.

Theorem (**Well-Posedness of the IVP problem**).
If $f(t, y)$ Satisfies the Lipschitz Condition, then the IVP is well-posed.

2 The Euler Method

One of the simplest methods for solving the IVP is the classical Euler method.

The method is derived from the Taylor Series expansion of the function $y(t)$.

The function $y(t)$ has the following Taylor series expansion of order n at $t = t_{i+1}$:

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2!} y''(t_i) + \cdots + \frac{(t_{i+1} - t_i)^n}{n!} y^{(n)}(t_i) + \frac{(t_{i+1} - t_i)^{n+1}}{(n+1)!} y^{n+1}(\xi_i),$$ where ξ_i is in (t_i, t_{i+1}).

Substitute $h = t_{i+1} - t_i$. Then

Taylor Series Expansion under n of $y(t)$

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i).$$

For $n = 1$, this formula reduces to

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi).$$

The term $= \dfrac{h^2}{2!} y^{(2)}(\xi_i)$ is call the **remainder term**.

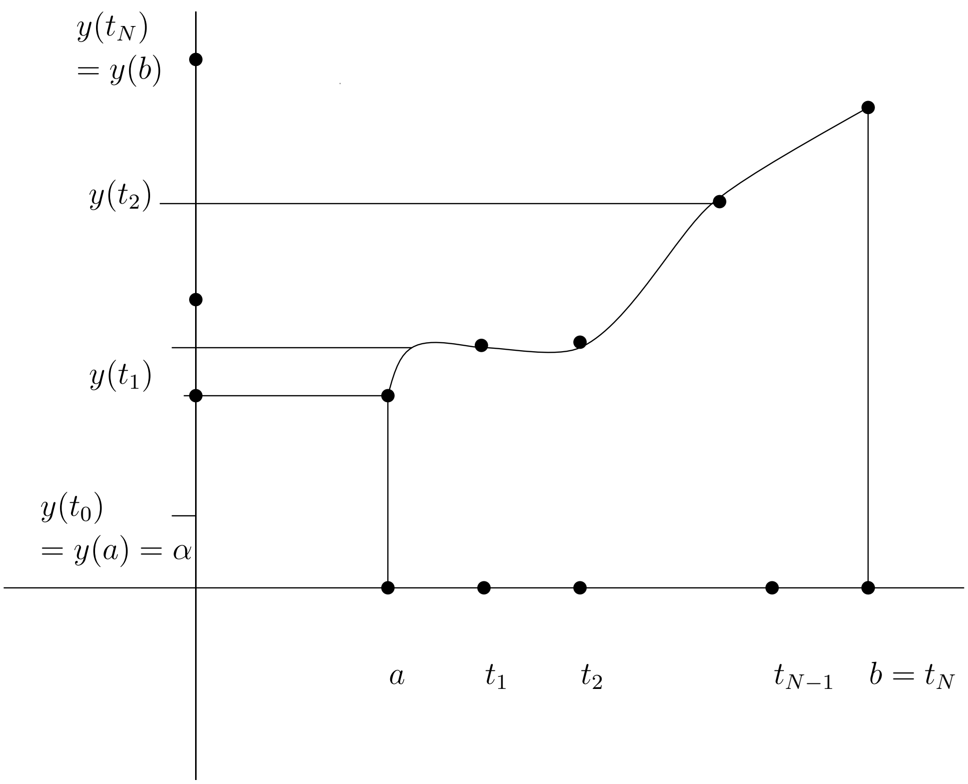
Neglecting the remainder term, we have

Euler's Method

$$\begin{aligned} y_{i+1} &= y_i + hy'(t_i) \\ &= y_i + hf(t_i, y_i), \end{aligned} \qquad i = 0, 1, 2, \cdots, N-1$$

This formula is known as the **Euler method** and now can be used to approximate $y(t_{i+1})$.

Geometrical Interpretation



Algorithm:	**Euler's Method for IVP**
Input:	(i). The function $f(t, y)$ (ii). The end points of the interval $[a, b]$: a and b (iii). The initial value: $\alpha = y(t_0) = y(a)$
Output:	Approximations y_{i+1} of $y(t_i + 1)$, $i = 0, 1, \cdots, N-1$.
Step 1.	**Initialization:** Set $t_0 = a, y_0 = y(t_0) = y(a) = \alpha$. and $N = \dfrac{b-a}{h}$.
Step 2.	For $i = 0, 1, \cdots, N-1$ do Compute $y_{i+1} = y_i + hf(t_i, y_i)$ End

Example: $y' = t^2 + 5, \qquad 0 \leq t \leq 1.$

$$y(0) = 0, \ h = 0.25$$

The points of subdivisions are: $t_0 = 0, t_1 = 0.25, t_2 = 0.50, t_3 = 0.75$ and $t_4 = 1$.

$\boldsymbol{i = 0}$: $t_1 = t_0 + h = 0.25$

$$y_1 = y_0 + hf(t_0, y_0) = 0 + .25(5) = 1.25 \ (\textbf{exact value} \text{ of } y(1): \ 1.2552)$$

$\boldsymbol{i = 1}$: $t_2 = t_1 + h = 0.50$

$$\begin{aligned} y_2 &= y_1 + hf(t_1, y_1) \\ &= 1.25 + 0.25(t_1^2 + 5) = 1.25 + 0.25((0.25)^2 + 5) \\ &= 2.5156 \ (\textbf{exact value} \text{ of } y(2): \ 2.5417) \end{aligned}$$

$\boldsymbol{i = 2}$: $t_3 = t_2 + h = 0.75$

$$\begin{aligned} y_3 &= y_2 + hf(t_2, y_2) \\ &= 2.5156 + .25((.5)^2 + 5) = 3.8281 \ (\textbf{exact value} \text{ of } y(3) : 3.8906) \end{aligned}$$

Note: The exact values above are correct up to 4 decimal digits.

Example: $y' = t^2 + 5, \ 0 \leq t \leq 2,$

$$y(0) = 0, h = 0.5$$

So, the points of subdivisions are: $t_0 = 0,\ t_1 = 0.5,\ t_2 = 1,\ t_3 = 1.5,\ t_4 = 2.$

We compute y_1, y_2, y_3, and y_4, which are, respectively, approximations to $y(0.5), y(1), y(1.5)$, and $y(2)$.

$i = 0:$ $$y_1 = y_0 + hf(t_0, y_0) = y(0) + hf(0,0) = 0 + 0.5 \times 5 = 2.5$$

$$(\textbf{exact Value} = 2.5417).$$

$i = 1:$ $$y_2 = y_1 + hf(t_1, y_1) = 2.5 + 0.5((0.5)^2 + 5) = 5.1250$$

$$(\textbf{exact Value} = 5.3333).$$

$i = 2:$ $$y_3 = y_2 + hf(t_2, y_2) = 5.1250 + 0.5(t_2^2 + 5) = 5.1250 + 0.5(1.5) = 8.1250$$

$$(\textbf{exact Value} = 8.6250)$$

The Errors in Euler's Method

The approximations obtained by a numerical method to solve the IVP are usually subjected to three types of errors:

- **Local Truncation Error**
- **Global Truncation Error**
- **Round-off Error**

Definition. The **local truncation error** is the error made at a **single step** due to the truncation of the series used to solve the problem.

Definition. The **global truncation error** is the truncation error at any step, that is, the total of the accumulative single-step truncation errors at previous steps.

Recall that the Euler Method was obtained by truncating the Taylor series

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots$$

after two terms. Thus, in obtaining Euler's method, the first term neglected was $\frac{h^2}{2}y''(t)$.

So the **local error in Euler's method is:** $E_L = \frac{h^2}{2}y''(\xi_i)$,
where ξ_i lies between t_i and t_{i+1}. In this case, we say **that the local error is of order h^2, written as $O(h^2)$.**

On the other hand, the *global truncation error* is of order $h : O(h)$, as can be seen from the following theorem.

Denote the global error at Step i by E_i, that is, $E_i = y(t_i) - y_i$.

Below we give a bound for this error assuming that certain properties of the derivatives of the solution are known. The proof of the result can be found in the book by Gear [*Numerical Initial Value Problems in Ordinary Differential Equations, Prentice Hall, Inc., (1971)*]

Theorem: (Global Error Bound for the Euler Method)

Let $y(t)$ be the unique solution of the IVP: $y' = f(t, y); y(a) = \alpha$.

$$a \le t \le b, -\infty < y < \infty,$$

Let L and M be two numbers such that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| \le L, \text{ and } ||y''(t)|| \le M \text{ in } [a,b].$$

Then the global error E_i at $t = t_i$ satisfies

$$|E_i| = \left|y(t_i) - y_i\right| \le \frac{hM}{2L}(e^{L(t_i - a)} - 1).$$

Thus, The global error bound for Euler's method depends upon h, whereas the local error depends upon h^2.

Remark. Since the exact solution $y(t)$ of the IVP is not known, the above bound may not be of practical importance as far as knowing how large the error can be a priori. However, from this error bound, we can say that the *Euler method can be made to converge faster by decreasing the step-size.* Furthermore, if the equalities, L and M of the above theorem can be found, then we can determine what step-size will be needed to achieve a certain accuracy, as the following example shows.

Example: $\dfrac{dy}{dt} = \dfrac{t^2+y^2}{2}, y(0) = 0$

$0 \le t \le 1,\ -1 \le y(t) \le 1.$

Determine how small the step-size should be so that the error

does not exceed $\epsilon = 10^{-4}$.

Since $f(t,y) = \dfrac{t^2+y^2}{2}$, we have

$$\frac{\partial f}{\partial y} = y$$

Thus, $\left|\dfrac{\partial f}{\partial y}\right| \le 1$ for all y, giving $L = 1$.

To find M, we compute the second-derivative of $y(t)$ as follows:

$$\begin{aligned}
y' &= \frac{dy}{dt} = f(t,y)(\text{Given}) \\
\text{By implicit differentiation, } y'' &= \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} \\
&= t + \left(\frac{t^2+y^2}{2}\right) y = t + \frac{y}{2}(t^2+y^2) \\
\text{So, } |y''(t)| &= |t + \frac{y}{2}(t^2+y^2)| \le 2,\ \text{for } 1 \le y \le 1. \\
\text{Thus, } & M = 2, \\
\text{and } & |E_i| = |y(t_i) - y_i| \le \frac{2h}{2L}(e^{(t_i)} - 1) = h(e^{(t_i)} - 1) = h(e-1).
\end{aligned}$$

Now, for the error not to exceed 10^{-4}, we must have:

$h(e-1) < 10^{-4}$ or $h < \dfrac{10^{-4}}{e-1} \approx 5.8198 \times 10^{-5}$.

3 High-order Taylor Methods

Recall that the Taylor's series expansion of $y(t)$ of degree n is given by

$$y(t_{i+1}) \quad = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

Now,

$$(i) \quad y'(t) = f(t, y(t)) \text{ (given)}.$$
$$(ii) \quad y''(t) = f'(t, y(t)).$$
$$(iii) \quad y^{(i)}(t) = f^{(i-1)}(t, y(t)), i = 1, 2, \ldots, n.$$

Thus,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(t_i, y_i) + \frac{h^n}{n!}f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!}f^{(n-1)}(\xi_i, y(\xi_i)$$

$$= y(t_i) + h\left[f(t_i, y(t_i)) + \frac{h}{2}f'(t_i, y(t_i)) + \cdots + \frac{h^{n-1}}{n!}f^{n-1}(t_i, y(t_i))\right] + \text{ Remainder Term}$$

Neglecting the remainder term the above formula can be written in compact form as follows:

$y_{i+1} = y_i + hT_k(t_i, y_i),\ i = 0, 1, \cdots, N-1$, where $T_k(t_i.y_i)$ is defined by:

$$T_k(t_i, y_1) = f(t_i, y_i) + \frac{h}{2}f'(t_i, y_i) + \cdots + \frac{h^{k-1}}{k!}f^{(k-1)}(t_i,\ y_i)$$

So, if we truncate the Taylor Series after $(k+1)$ terms and use the truncated series to obtain the approximating of y_{1+1} of $y(t_{i+1})$, we have the following **of k-th order Taylor's algorithm for the IVP**.

Taylor's Algorithm of order k for IVP

Input: (i) The function $f(t, y)$

(ii) The end points: a and b

(iii) The initial value: $\alpha = y(t_o) = y(a)$

(iv) The order of the algorithm: k

(v) The step size: h

Step 1 **Initialization:** $t_0 = a,\ y_0 = \alpha,\ N = \dfrac{b-a}{h}$

Step 2. For $i = \ldots, N-1$ *do*

2.1 Compute $T_k(t_i, y_i) = f(t_i, y_i) + \dfrac{h}{2} f'(t_i, y_i) + \ldots + \dfrac{h^{k-1}}{k!} f^{(k-1)}(t_i, y_i)$

2.2 Compute $y_{i+1} = y_i + hT_k(t_i, y_i)$

End

Note: With $k = 1$, the above formula for y_{i+1}, reduces to Euler's method.

Example:

$y' = y - t^2 + 1, \quad 0 \le t \le 2,\ y(0) = 0.5, h = 0 \cdot 2.$

$$
\begin{aligned}
f(t, y(t)) &= y - t^2 + 1 \text{ (Given).} \\
f'(t, y(t)) &= \frac{d}{dt}(y - t^2 + 1) = y' - 2t \\
&= y - t^2 + 1 - 2t \\
f''(t, y(t)) &= \frac{d}{dt}(y - t^2 + 1 - 2t) = y - t^2 + 1 - 2t - 2 \qquad = \quad y - t^2 - 2t - 1 \\
\text{so,} & \\
y(0.2) \simeq y_1 &= y_0 + hf(t_0, y(t_0)) + \frac{h^2}{2} f'(t_0, y(t_0)) \\
&= 0.5 + 0.2 \times 1.5 + \frac{(0.2)^2}{2}(0.5 + 1) = 2.2300\textbf{check}. \\
y(0.4) &\simeq y_2 = 1.215800
\end{aligned}
$$

4 Runge-Kutta Methods

- The Euler's method is the simplest to implement; however, even for a reasonable accuracy the step-size h needs to be very small.

- The difficulties with higher order Taylor's series methods are that the derivatives of higher orders of $f(t, y)$ need to be computed, which are very often difficult to compute/ needles to say that if $f(t, y)$ is not explicity known in many areas.

The Runge-Kutta methods aim at achieving the accuracy of higher order Taylor series methods without computing the higher order derivatives.

We first develop the simplest one: **The Runge-Kutta Methods of order 2.**

The Runge-Kutta Methods of order 2

Suppose that we want an expression of the approximation y_{i+1} in the form:

$$y_{i+1} = y_i + \alpha_1 k_1 + \alpha_2 k_2, \tag{4.1}$$

$$\text{where } k_1 = hf(t_i, y_i), \tag{4.2}$$

and

$$k_2 = hf(t_i + \alpha h,\ y_i + \beta k_1). \tag{4.3}$$

The constants α_1 and α_2 and α and β are to be chosen so that the formula is as accurate as the Taylor's Series Method with $n = 1$.

To develop the method we need an important result from Calculus: **Taylor's series for function to two variables.**

Taylor's Theorem for Function of Two Variables

Let $f(t, y)$ and its partial derivatives of orders up to $(n+1)$ are continuous in the domain $D = \{(t, y)|a \leq t \leq b,\ c \leq y \leq d\}$.

Then

$$f(t, y) = f(t_0, y_0) + \left[(t - t_0)\frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(t_0, y_0)\right] + \cdots$$

$$+\left[\frac{1}{n!}\sum_{h=0}^{n}\binom{n}{i}(t - t_0)^{h-i}(y - y_0)^i\frac{\partial^n f}{\partial t^{n-1}\partial y^i}(t_0, y_0)\right] + R_n(t, y),$$

where $R_n(t, y)$ is the remainder after n terms and involves the partial derivative of order $n + 1$.

Using the above theorem with $n = 1$, we have

$$f(t_i + \alpha h,\ y_i + \beta k_1) = f(t_i, y_i) + \alpha h\frac{\partial f}{\partial t}(t_i, y_i) + \beta k_1\frac{\partial f}{\partial y}(t_i, y_i) \tag{4.4}$$

From (4.4) and (4.3), we obtain

$$\frac{k_2}{h} = f(t_i, y_i) + \alpha h\frac{\partial f}{\partial t}(t_i.y_i) + \beta k_1\frac{\partial f}{\partial y}(t_i, y_i). \tag{4.5}$$

Again, substituting the value of k_1 from (4.2) and k_2 from (4.3) in (4.1) we get (after some rearrangment):

$$
\begin{aligned}
y_{i+1} &= y_i + \alpha_1 h f(t_i, y_i) + \alpha_1 h \left[f(t_i, y_i) + \alpha h \frac{\partial f}{\partial t}(t_i, y_i) + \beta h f(t_i, y_i) \frac{\partial f}{\partial y}(t_i, y_i) \right] \\
&= y_i + (\alpha_1 + \alpha_2) h f(t_i, y_i) + \alpha_2 h^2 \left[\alpha \frac{\partial f}{\partial t}(t_i, y_i) + \beta f(t_i, y_i) \frac{\partial f}{\partial y}(t_i, y_i) \right]
\end{aligned} \tag{4.6}
$$

Also, note that $y(t_{i+1}) = y(t_i) + hf(t_i, y_i) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_i, y_i) + f(t_i, y_i)\frac{\partial f}{\partial y}(t_i, y_i)\right)$ + higher order terms.

So, neglecting the higher order terms, we can write

$$
y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_i, y_i) + f\frac{\partial f}{\partial y}(t_i, y_i)\right). \tag{4.7}
$$

If we want (4.6) and (4.7) to agree for numerical approximations, then we must have

- $\alpha_1 + \alpha_2 = 1$ (comparing the coefficients of $hf(t_i, y_i)$).
- $\alpha_2 \alpha = \frac{1}{2}$ (comparing the coefficients of $h^2 \frac{\partial f}{\partial t}(t_i, y_i)$.
- $\alpha_2 \beta = \frac{1}{2}$ (comparing the coefficents of $h^2 f(t_i, y_i)\frac{\partial f}{\partial y}(t_i y_i)$.

Since the number of unknowns here exceeds the number of equations, there are infinitely many possible solutions. The simplest solution is:

$$
\boxed{\alpha_1 = \alpha_2 = \frac{1}{2}, \ \alpha = \beta = 1}.
$$

With these choices we can generate y_{i+1} from y_i as follows. The process is known as the **Modified Euler's Method**.

Generating y_{i+1} from y_i in Modified Euler's Method

$$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2),$$

where $k_1 = hf(t_i, y_i)$

$$k_2 = hf(t_i + h,\ y_i + k_1).$$

or

$$y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_i + h,\ y_i + hf(t_i, y_i)\right]$$

Algorithm: The Modified Euler Method

Inputs: The given function: $f(t, y)$
The end points of the interval: a and b
The step-size: h
The initial value $y(t_0) = y(a) = \alpha$

Outputs: Approximations y_{i+1} of $y(t_{i+1}) = y(t_0 + ih)$, $i = 0, 1, 2, \cdots, N-1$

Step 1 **(Initialization)**
Set $t_0 = a,\ y_0 = y(t_0) = y(a) = \alpha$
$N = \dfrac{b-a}{h}$

Step 2 For $i = 0, 1, 2, \cdots, N-1$ do
Compute $k_1 = hf(t_i, y_i)$
Compute $k_2 = hf(t_i + h, y_i + k_1)$
Compute $y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$.

End

Local Error in the Modified Euler Method

Since in deriving the modified Euler method, we neglected the terms involving h^3 and higher powers of h, the **local error for this method is** $O(h^3)$. **Thus with the Modified Euler**

method, we will be able to use larger step-size h than the Euler Method to obtain the same accuracy.

Example: $y' = t + y,\ y(0) = 1$

$h = 0.01, y_0 = y(0) = 1.$

$$\begin{aligned} i = 0: \quad y_1 &= y_0 + \frac{1}{2}(k_1 + k_2) \\ k_1 &= hf(t_0, y_0) = 0.01(0+1) = 0.01 \\ k_2 &= hf(t_0 + h,\ y_0 + k_1) = 0.01 \times f(0.01,\ 1 + 0.01) \\ &\qquad\quad = 0.01 \times (0.01 + 1.01) = 0.01 \times 1.02 = 0.0102 \end{aligned}$$

$$y(0.01) \approx y_1 = 1 + \frac{1}{2}(0.01 + 0.0102) = 1.0101$$

$$\begin{aligned} i = 1: \quad y_2 &= y_1 + \frac{1}{2}(k_1 + k_2) \\ k_1 &= hf(t_1, y_1) \\ &= 0.01 \times f(0.01, 1.0101) = 0.01 \times (0.01 + 1.0101) \\ &= 0.0102 \\ \\ k_2 &= hf(t_1 + h,\ y_1 + k_1) \\ &= 0.01 \times f(0.02,\ 1.0101 + 0.0102) = 0.01 \times (0.02 + 1.0203) \\ &= -0.0104 \end{aligned}$$

$$y(0.02) \approx y_2 = 1.0101 + \frac{1}{2}(0.0102 + 0.0104) = 1.0204$$

The Midpoint and Heun's Methods

In deriving the modified Euler's Method, we have considered only one set of possible values of $\alpha_1, \alpha_2, \alpha_1$ and β. We will now consider two more sets of values.

- $\alpha = 0,\ \alpha_2 = 1,\ \alpha = \beta = \frac{1}{2}.$

This gives us the **Midpoint Method**.

The Midpoint Method

$y_{i+1} = y_i + k_2$

where $k_1 = hf(t_i, y_i)$

$$k_2 = hf\left(t_i + \frac{h}{2},\ \ y_i + \frac{k_1}{2}\right)$$

or

$$y_{i+1} = y_i + hf\left(t_i + \frac{h}{2},\ \ y_i + \frac{h}{2} f(t_i, y_i)\right), i = 0, 1, \ldots, N-1.$$

- $\alpha_1 = \frac{1}{4},\ \beta_1 = \frac{3}{4},\ \alpha = \beta = \frac{2}{3}$

Then we have **Heun's Method**.

Heun's Method

$$y_{i+1} = y_i + \frac{1}{4}k_1 + \frac{3}{4}k_2$$

where $k_1 = hf(t_i, y_i)$

$$k_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}k_1\right)$$

or

$$y_{i+1} = y_i + \frac{h}{4} f(t_i, y_i) + \frac{3h}{4} f\left(t_i + \frac{2}{3}h, y_i + \frac{2h}{3} f(t_i, y_i)\right), i = 0, 1, \cdots, N-1$$

Heun's Method and the **Modified Euler's Method** are classified as the **Runge-Kutta methods of order 2**.

The Runge-Kutta Method of order 4.

A method very widely used in practice is the Runge-Kutta method of order 4. It's derivation is complicated. We will just state the method, without proof.

Algorithm: The Runge-Kutta Method of Order 4

Inputs: $f(t, y)$ - the given function
a, b - the end points of the interval
α - the initial value $y(t_0)$
h - the step size

Outputs: The approximations y_{i+1} of $y(t_{i+1})$, $i = 0, 1, \cdots, N-1$

Step 1: **(Initialization)**
Set $t_0 = a$, $y_0 = y(t_0) = y(a) = \alpha$
$N = \dfrac{b-a}{h}$.

Step 2: **(Computations of the Runge-Kutta Coefficients)**
For $i = 0, 1, 2, \cdots, n$ do

$$k_1 = hf(t_i, y_i)$$

$$k_2 = hf(t_i + \frac{h}{2}, \ \ y_i + \frac{1}{2}k_1)$$

$$k_3 = hf(t_i + \frac{h}{2}, \ \ y_i + \frac{1}{2}k_2)$$

$$k_4 = hf(t_i + h, \ \ y_i + k_3)$$

Step 3: **(Computation of the Approximate Solution)**
Compute: $y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

The Local Truncation Error: The local truncation error of the Runge-Kutta Method of order 4 is $O(h^5)$.

Example:

$$y' = t + y, \quad y(0) = 1$$

$$h = 0.01$$

Let's complete $y(0.01)$ using the Runge-Kutta Method of order 4.

$i = 0$

$y(0.01) \approx y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = hf(t_0, y_0) = 0.01f(0,1) = 0.01 \times 1 = 0.01$.

$k_2 = hf(t_0 + \frac{h}{2}, \;\; y_0 + \frac{k_1}{2}) = 0.01f\left(\frac{0.01}{2}, 1 + \frac{0.01}{2}\right) = 0.01\left[\frac{0.01}{2} + \frac{1+0.01}{2}\right] = 0.0101.$

$k_3 = hf\left(t_0 + \frac{h}{2}, \;\; y_0 + \frac{k_2}{2}\right) = h\left(t_0 + \frac{h}{2} + y_0 + \frac{k_2}{2}\right) = 0.0101005.$

$k_4 = hf(t_0 + h, y_0 + k_3) = h(t_0 + h + y_0 + k_3) = 0.01020100$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.010100334$

and so on.