3.6 The Chain Rule

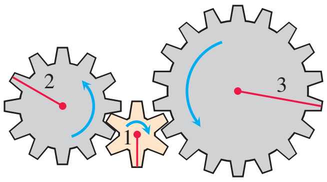


FIGURE 3.25 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

How do we differentiate $F(x) = \sin(x^2 - 4)$? This function is the composite $f \circ g$ of two functions $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$ that we know how to differentiate. The answer, given by the *Chain Rule*, says that the derivative is the product of the derivatives of f and g. We develop the rule in this section.

Derivative of a Composite Function

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composite of the functions $y = \frac{1}{2}u$ and $u = 3x$. We have

$$\frac{dy}{dx} = \frac{3}{2}, \qquad \frac{dy}{du} = \frac{1}{2}, \qquad \text{and} \qquad \frac{du}{dx} = 3.$$

Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see in this case that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. If $y = f(u)$ changes half as fast as u and $u = g(x)$ changes three times as fast as x, then we expect y to change $3/2$ times as fast as x. This effect is much like that of a multiple gear train (Figure 3.25). Let's look at another example.

EXAMPLE 1 The function

$$y = (3x^2 + 1)^2$$

is the composite of $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \qquad \text{Substitute for } u \\ &= 36x^3 + 12x. \end{aligned}$$

Calculating the derivative from the expanded formula $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$ gives the same result:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x. \end{aligned}$$

■

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Figure 3.26).

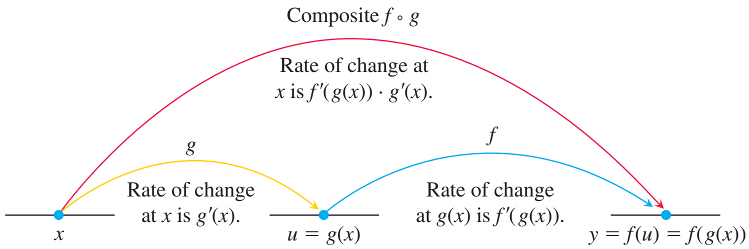


FIGURE 3.26 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

A Proof of One Case of the Chain Rule:

Let Δu be the change in u when x changes by Δx, so that

$$\Delta u = g(x + \Delta x) - g(x).$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u).$$

If $\Delta u \neq 0$, we can write the fraction $\Delta y / \Delta x$ as the product

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \tag{1}$$

and take the limit as $\Delta x \rightarrow 0$:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \qquad \text{(Note that } \Delta u \to 0 \text{ as } \Delta x \to 0 \text{ since } g \text{ is continuous.)} \\
&= \frac{dy}{du} \cdot \frac{du}{dx}.
\end{aligned}$$

The problem with this argument is that if the function $g(x)$ oscillates rapidly near x, then Δu can be zero even when $\Delta x \neq 0$, so the cancelation of Δu in Equation (1) would be invalid. A complete proof requires a different approach that avoids this problem, and we give one such proof in Section 3.11. ■

EXAMPLE 2 An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Solution We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{du}\cdot\frac{du}{dt}\\ &= -\sin(u)\cdot 2t \qquad \frac{dx}{du} \text{ evaluated at } u\\ &= -\sin(t^2+1)\cdot 2t\\ &= -2t\sin(t^2+1).\end{aligned}$$

Ways to Write the Chain Rule

$$(f\circ g)'(x) = f'(g(x))\cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = f'(g(x))\cdot g'(x)$$

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}$$

"Outside-Inside" Rule

A difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So it sometimes helps to think about the Chain Rule using functional notation. If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x))\cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

EXAMPLE 3 Differentiate $\sin(x^2 + e^x)$ with respect to x.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin\underbrace{(x^2+e^x)}_{\text{inside}} = \cos\underbrace{(x^2+e^x)}_{\substack{\text{inside}\\\text{left alone}}}\cdot\underbrace{(2x+e^x)}_{\substack{\text{derivative of}\\\text{the inside}}}.$$

EXAMPLE 4 Differentiate $y = e^{\cos x}$.

Solution Here the inside function is $u = g(x) = \cos x$ and the outside function is the exponential function $f(x) = e^x$. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x}\sin x.$$

Generalizing Example 4, we see that the Chain Rule gives the formula

$$\boxed{\frac{d}{dx}e^u = e^u\frac{du}{dx}.}$$

For example,

$$\frac{d}{dx}(e^{kx}) = e^{kx}\cdot\frac{d}{dx}(kx) = ke^{kx}, \qquad \text{for any constant } k$$

and

$$\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2}\cdot\frac{d}{dx}(x^2) = 2xe^{x^2}.$$

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

HISTORICAL BIOGRAPHY
Johann Bernoulli
(1667–1748)

EXAMPLE 5 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$

The Chain Rule with Powers of a Function

If f is a differentiable function of u and if u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}.$$

If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 6 The Power Chain Rule simplifies computing the derivative of a power of an expression.

(a)
$$\begin{aligned} \frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) && \text{Power Chain Rule with } u = 5x^3 - x^4, n = 7 \\ &= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3) \\ &= 7(5x^3 - x^4)^6(15x^2 - 4x^3) \end{aligned}$$

(b)
$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{3x - 2}\right) &= \frac{d}{dx}(3x - 2)^{-1} \\ &= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2) && \text{Power Chain Rule with } u = 3x - 2, n = -1 \\ &= -1(3x - 2)^{-2}(3) \\ &= -\frac{3}{(3x - 2)^2} \end{aligned}$$

In part (b) we could also find the derivative with the Derivative Quotient Rule.

(c)
$$\begin{aligned} \frac{d}{dx}(\sin^5 x) &= 5\sin^4 x \cdot \frac{d}{dx}\sin x && \text{Power Chain Rule with } u = \sin x, n = 5, \text{ because } \sin^n x \text{ means } (\sin x)^n, n \neq -1. \\ &= 5\sin^4 x \cos x \end{aligned}$$

(d) $$\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}} \cdot \frac{d}{dx}\left(\sqrt{3x+1}\right)$$

$$= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3$$ Power Chain Rule with $u = 3x + 1$, $n = 1/2$

$$= \frac{3}{2\sqrt{3x+1}} e^{\sqrt{3x+1}}$$ ■

EXAMPLE 7 In Section 3.2, we saw that the absolute value function $y = |x|$ is not differentiable at $x = 0$. However, the function is differentiable at all other real numbers, as we now show. Since $|x| = \sqrt{x^2}$, we can derive the following formula:

Derivative of the Absolute Value Function

$$\frac{d}{dx}(|x|) = \frac{x}{|x|}, \quad x \neq 0$$

$$= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$$\frac{d}{dx}(|x|) = \frac{d}{dx}\sqrt{x^2}$$

$$= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2)$$ Power Chain Rule with $u = x^2$, $n = 1/2$, $x \neq 0$

$$= \frac{1}{2|x|} \cdot 2x$$ $\sqrt{x^2} = |x|$

$$= \frac{x}{|x|}, \quad x \neq 0.$$ ■

EXAMPLE 8 Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

Solution We find the derivative:

$$\frac{dy}{dx} = \frac{d}{dx}(1 - 2x)^{-3}$$

$$= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x)$$ Power Chain Rule with $u = (1 - 2x)$, $n = -3$

$$= -3(1 - 2x)^{-4} \cdot (-2)$$

$$= \frac{6}{(1 - 2x)^4}.$$

At any point (x, y) on the curve, the coordinate x is not $1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

which is the quotient of two positive numbers. ■

EXAMPLE 9 The formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians where $x°$ is the size of the angle measured in degrees.

By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Figure 3.27. Similarly, the derivative of $\cos(x°)$ is $-(\pi/180)\sin(x°)$.

The factor $\pi/180$ would compound with repeated differentiation, showing an advantage for the use of radian measure in computations. ■

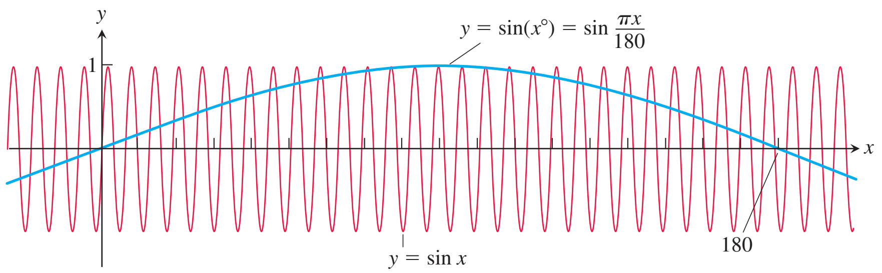


FIGURE 3.27 The function $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 9).

Exercises 3.6

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = e^{-x}$
5. $y = \sqrt{u}, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = \pi x^2$
8. $y = -\sec u, \quad u = \frac{1}{x} + 7x$

In Exercises 9–22, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \frac{x}{7}\right)^{-7}$
12. $y = \left(\frac{\sqrt{x}}{2} - 1\right)^{-10}$
13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
14. $y = \sqrt{3x^2 - 4x + 6}$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \frac{1}{x}\right)$
17. $y = \tan^3 x$
18. $y = 5\cos^{-4} x$
19. $y = e^{-5x}$
20. $y = e^{2x/3}$
21. $y = e^{5-7x}$
22. $y = e^{(4\sqrt{x}+x^2)}$

Find the derivatives of the functions in Exercises 23–50.

23. $p = \sqrt{3 - t}$
24. $q = \sqrt[3]{2r - r^2}$
25. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$
26. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$
27. $r = (\csc\theta + \cot\theta)^{-1}$
28. $r = 6(\sec\theta - \tan\theta)^{3/2}$
29. $y = x^2\sin^4 x + x\cos^{-2} x$
30. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x$
31. $y = \frac{1}{18}(3x - 2)^6 + \left(4 - \frac{1}{2x^2}\right)^{-1}$
32. $y = (5 - 2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$
33. $y = (4x + 3)^4(x + 1)^{-3}$
34. $y = (2x - 5)^{-1}(x^2 - 5x)^6$
35. $y = xe^{-x} + e^{x^3}$
36. $y = (1 + 2x)e^{-2x}$
37. $y = (x^2 - 2x + 2)e^{5x/2}$
38. $y = (9x^2 - 6x + 2)e^{x^3}$
39. $h(x) = x\tan\left(2\sqrt{x}\right) + 7$
40. $k(x) = x^2\sec\left(\frac{1}{x}\right)$
41. $f(x) = \sqrt{7 + x\sec x}$
42. $g(x) = \frac{\tan 3x}{(x + 7)^4}$
43. $f(\theta) = \left(\frac{\sin\theta}{1 + \cos\theta}\right)^2$
44. $g(t) = \left(\frac{1 + \sin 3t}{3 - 2t}\right)^{-1}$
45. $r = \sin(\theta^2)\cos(2\theta)$
46. $r = \sec\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)$
47. $q = \sin\left(\frac{t}{\sqrt{t + 1}}\right)$
48. $q = \cot\left(\frac{\sin t}{t}\right)$
49. $y = \cos\left(e^{-\theta^2}\right)$
50. $y = \theta^3 e^{-2\theta}\cos 5\theta$

In Exercises 51–70, find dy/dt.

51. $y = \sin^2(\pi t - 2)$
52. $y = \sec^2 \pi t$
53. $y = (1 + \cos 2t)^{-4}$
54. $y = (1 + \cot(t/2))^{-2}$
55. $y = (t\tan t)^{10}$
56. $y = (t^{-3/4}\sin t)^{4/3}$
57. $y = e^{\cos^2(\pi t - 1)}$
58. $y = \left(e^{\sin(t/2)}\right)^3$
59. $y = \left(\frac{t^2}{t^3 - 4t}\right)^3$
60. $y = \left(\frac{3t - 4}{5t + 2}\right)^{-5}$
61. $y = \sin(\cos(2t - 5))$
62. $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$
63. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$
64. $y = \frac{1}{6}\left(1 + \cos^2(7t)\right)^3$
65. $y = \sqrt{1 + \cos(t^2)}$
66. $y = 4\sin\left(\sqrt{1 + \sqrt{t}}\right)$
67. $y = \tan^2(\sin^3 t)$
68. $y = \cos^4(\sec^2 3t)$
69. $y = 3t(2t^2 - 5)^4$
70. $y = \sqrt{3t + \sqrt{2 + \sqrt{1 - t}}}$

Second Derivatives

Find y'' in Exercises 71–78.

71. $y = \left(1 + \frac{1}{x}\right)^3$
72. $y = \left(1 - \sqrt{x}\right)^{-1}$
73. $y = \frac{1}{9}\cot(3x - 1)$
74. $y = 9\tan\left(\frac{x}{3}\right)$
75. $y = x(2x + 1)^4$
76. $y = x^2(x^3 - 1)^5$
77. $y = e^{x^2} + 5x$
78. $y = \sin(x^2 e^x)$

Finding Derivative Values

In Exercises 79–84, find the value of $(f \circ g)'$ at the given value of x.

79. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

80. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1 - x}, \quad x = -1$

81. $f(u) = \cot \frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

82. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

83. $f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

84. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

85. Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is y' at $x = 2$?

86. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?

87. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a. $2f(x), \quad x = 2$ **b.** $f(x) + g(x), \quad x = 3$
c. $f(x) \cdot g(x), \quad x = 3$ **d.** $f(x)/g(x), \quad x = 2$
e. $f(g(x)), \quad x = 2$ **f.** $\sqrt{f(x)}, \quad x = 2$
g. $1/g^2(x), \quad x = 3$ **h.** $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

88. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at the given value of x.

a. $5f(x) - g(x), \quad x = 1$ **b.** $f(x)g^3(x), \quad x = 0$
c. $\frac{f(x)}{g(x) + 1}, \quad x = 1$ **d.** $f(g(x)), \quad x = 0$
e. $g(f(x)), \quad x = 0$ **f.** $(x^{11} + f(x))^{-2}, \quad x = 1$
g. $f(x + g(x)), \quad x = 0$

89. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

90. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Theory and Examples

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 91 and 92.

91. Find dy/dx if $y = x$ by using the Chain Rule with y as a compsite of

a. $y = (u/5) + 7$ and $u = 5x - 35$
b. $y = 1 + (1/u)$ and $u = 1/(x - 1)$.

92. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composite of

a. $y = u^3$ and $u = \sqrt{x}$
b. $y = \sqrt{u}$ and $u = x^3$.

93. Find the tangent to $y = ((x - 1)/(x + 1))^2$ at $x = 0$.

94. Find the tangent to $y = \sqrt{x^2 - x + 7}$ at $x = 2$.

95. a. Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.

b. Slopes on a tangent curve What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

96. Slopes on sine curves

a. Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.

b. Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.

c. For a given m, what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.

d. The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

97. Running machinery too fast Suppose that a piston is moving straight up and down and that its position at time t sec is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why some machinery breaks when you run it too fast.)

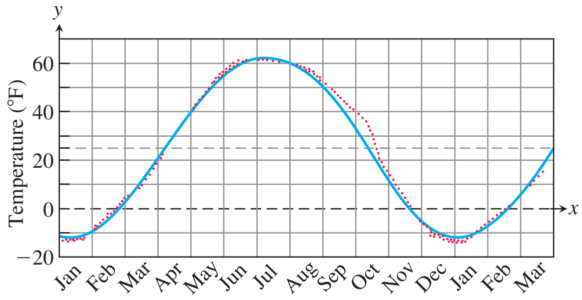
98. Temperatures in Fairbanks, Alaska The graph in the accompanying figure shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25$$

and is graphed in the accompanying figure.

a. On what day is the temperature increasing the fastest?

b. About how many degrees per day is the temperature increasing when it is increasing at its fastest?



99. Particle motion The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

100. Constant acceleration Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s m from its starting point. Show that the body's acceleration is constant.

101. Falling meteorite The velocity of a heavy meteorite entering Earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s km from Earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

102. Particle acceleration A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

103. Temperature and the period of a pendulum For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

104. Chain Rule Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

T **105. The derivative of sin 2x** Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x + h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

106. The derivative of cos(x²) Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos((x + h)^2) - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

Using the Chain Rule, show that the Power Rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 107 and 108.

107. $x^{1/4} = \sqrt{\sqrt{x}}$ **108.** $x^{3/4} = \sqrt{x\sqrt{x}}$

COMPUTER EXPLORATIONS

Trigonometric Polynomials

109. As the accompanying figure shows, the trigonometric "polynomial"

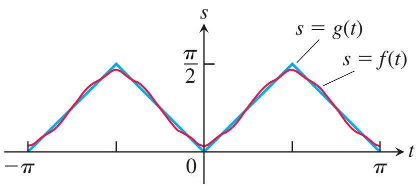
$$s = f(t) = 0.78540 - 0.63662\cos 2t - 0.07074\cos 6t - 0.02546\cos 10t - 0.01299\cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

a. Graph dg/dt (where defined) over $[-\pi, \pi]$.

b. Find df/dt.

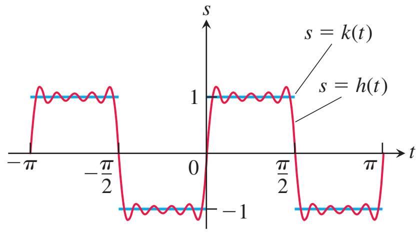
c. Graph df/dt. Where does the approximation of dg/dt by df/dt seem to be best? Least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.



110. (*Continuation of Exercise 109.*) In Exercise 109, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the trigonometric "polynomial"

$$s = h(t) = 1.2732\sin 2t + 0.4244\sin 6t + 0.25465\sin 10t + 0.18189\sin 14t + 0.14147\sin 18t$$

graphed in the accompanying figure approximates the step function $s = k(t)$ shown there. Yet the derivative of h is nothing like the derivative of k.



a. Graph dk/dt (where defined) over $[-\pi, \pi]$.

b. Find dh/dt.

c. Graph dh/dt to see how badly the graph fits the graph of dk/dt. Comment on what you see.

3.7 Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0, \qquad y^2 - x = 0, \qquad \text{or} \qquad x^2 + y^2 - 25 = 0.$$

(See Figures 3.28, 3.29, and 3.30.) These equations define an *implicit* relation between the variables x and y. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by *implicit differentiation*. This section describes the technique.

Implicitly Defined Functions

We begin with examples involving familiar equations that we can solve for y as a function of x to calculate dy/dx in the usual way. Then we differentiate the equations implicitly, and find the derivative to compare the two methods. Following the examples, we summarize the steps involved in the new method. In the examples and exercises, it is always assumed that the given equation determines y implicitly as a differentiable function of x so that dy/dx exists.

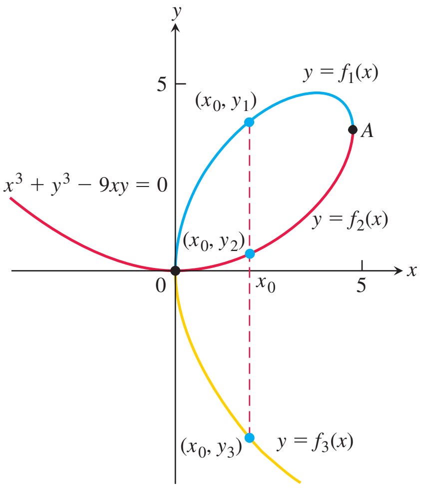


FIGURE 3.28 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. The curve can, however, be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium*, dates to Descartes in 1638.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 3.29). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx?

The answer is yes. To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

The Chain Rule gives $\frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}$.