Def A matrix $B_{n\times n}$ is said to be similar to a matrix $A_{n\times n}$ if there exists a nonsingular matrix $S_{n\times n}$ such that $B = S^{-1}AS$

ex The matrix $B = \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix}$ is similar to the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

for $S = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ show this

$$S^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

$$B \stackrel{?}{=} \boxed{S^{-1}AS}$$

$$\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \cdots = B$$

$\Rightarrow$ A, B are similar.

$\boxed{B = S^{-1}AS}$

$\boxed{A = SBS^{-1}}$

$\boxed{AS = SB}$

A is similar to B $\Leftrightarrow$ B is similar to A

A and B are similar.

note If $A_{n\times n}$ and $B_{n\times n}$ are similar matrices then:

1. $|A| = |B|$
2. trace(A) = trace(B)
3. Characteristic polynomial of A = characteristic polynomial of B
4. eigenvalues of (A) = eigenvalues of (B)

$|A| = 6 = (2)(3) - 0$

$|B| = 6 = (-1)(6) - (-2)(6) = -6 + 12$

$tr(A) = 2 + 3 = 5$

$tr(B) = -1 + 6 = 5$

eigenvalues of A = 2, 3

" " " B = 2, 3

$p(\lambda) = (2-\lambda)(3-\lambda)$

$= 6 - 5\lambda + \lambda^2$

Sec. 6.3 Diagonalization:

Def An $n\times n$ matrix A is said to be diagonalizable if there exists a nonsingular matrix X and a diagonal matrix D such that $\boxed{A = XDX^{-1}} \equiv \boxed{D = X^{-1}AX}$

We say that X diagonalizes A.

i.e $A_{n\times n}$ is diagonalizable if it is similar to a diagonal matrix

Thm If $\lambda_1, \lambda_2, \dots, \lambda_K$ are distinct eigenvalues of an $n\times n$ matrix A with corresponding eigenvectors $x_1, x_2, \dots, x_K$, then the vectors $x_1, x_2, \dots, x_K$ are Linearly independent.

Thm An $n\times n$ matrix A is diagonalizable ~~iff~~ if and only if A has n Linearly independent eigenvectors.

Note If the eigen values of (λ) are distinct then A is diagonalizable.

ex let $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

1. find the eigen values of A
2. find the eigen vectors of A
3. Is A diagonalizable, if yes find X, D that diagonalizes A
4. find $A^{(100)}$

Sol $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow |A - \lambda I| = 0$

$(2-\lambda)(-5-\lambda) - (-3)(2) = 0$

$-10 + \lambda^2 + 3\lambda + 6 = 0$

$\lambda^2 + 3\lambda - 4 = 0$

$(\lambda - 1)(\lambda + 4) = 0$

$\Downarrow \qquad \Downarrow$

$\lambda_1 = 1 \quad \lambda_2 = -4$

$\lambda_1 = 1 \Rightarrow$

$$\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 2 & -6 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = \alpha$

$x_1 = 3\alpha$

$\alpha \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ take the case $\alpha = 1 \Rightarrow x_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\boxed{2}$ $\lambda_2 = -4 \Rightarrow$

$$\begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & -3 & 0 \\ 2 & -1 & 0 \end{array}\right] \div 3 \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = \alpha$

$x_1 = \frac{1}{2}\alpha$

$\Rightarrow \alpha \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ take the case $\alpha = 2 \Rightarrow x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

---

note that A is diagonalizable because λ_1, λ_2 are distinct.

or because $\begin{vmatrix} 3 & \frac{1}{2} \\ 1 & 1 \end{vmatrix} \neq 0$

$X = \begin{bmatrix} 3 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$, ..., $\boxed{X = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}$, $X = \begin{bmatrix} 6 & 1 \\ 2 & 2 \end{bmatrix}$, ...

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$

$X^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$

H.W show that

$A = X D X^{-1}$

$\begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$

3

4) $A^{(100)} = (XDX^{-1})^{100}$

$A^{(100)} = X D^{100} X^{-1}$

$= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (1)^{100} & 0 \\ 0 & (-4)^{100} \end{bmatrix} \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4^{100} \end{bmatrix} \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$

$=$

In general: If A is diagonalizable

$A = XDX^{-1}$

$\Rightarrow A^k = X D^k X^{-1}$

$= X \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots \\ 0 & \lambda_2^k & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix} X^{-1}$

Ex

let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{bmatrix}$

note A and B have the same eigenvalues!

$\lambda_1 = 4$

$\lambda_2 = \lambda_3 = 2$ not distinct.

---

take A:

$(A - \lambda I) X = 0 \implies$

If ① $\lambda_1 = 4$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{cccc} x_1 & & x_3 & \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} & \left|\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] \end{array}$$

$x_2 = \alpha$

$x_3 = 0 \implies \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ take $\alpha = 1 \implies \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_1$

$x_1 = 0$

If ② $\lambda_2 = 2 \implies$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_3 = \alpha$

$x_2 = 0 \implies \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \implies \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ take the case $\alpha = 1 \implies \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3$

$x_1 = 0$

We have just two L.Ind. eigenvectors e_1, e_3

so A is <u>not diagonalizable</u> ~~B~~ i.e A is <u>defective</u>

take B:

$$\lambda_1 = 4 \Rightarrow \begin{bmatrix} -2 & 0 & 0 \\ -1 & 0 & 0 \\ -3 & 6 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & 6 & -2 & 0 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -3 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = \alpha$

$x_2 = \frac{1}{3}\alpha \Rightarrow \alpha \begin{bmatrix} 0 \\ \frac{1}{3} \\ 1 \end{bmatrix}$ take $\alpha = 3 \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$x_1 = 0$

If $\lambda_2 = 2$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \\ -3 & 6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} x_1 \\ \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_2 = \alpha$

$x_3 = \beta$

$x_1 = 2\alpha$

$$\begin{bmatrix} 2\alpha \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2\alpha \\ \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

take $\alpha = 1$, $\beta = 1$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>now</u> $X = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $|X| \neq 0 \Rightarrow x_1, x_2, x_3$ are L.Ind $\Rightarrow$ B is diagonalizable

Scanned with CamScanner

$B = X D X^{-1}$

$= \left[\begin{matrix} \cancel{0} & 2 & \cancel{0} \\ \cancel{1} & 1 & \cancel{0} \\ \cancel{3} & 0 & \cancel{1} \end{matrix}\right] [\quad]$

6

when $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$X = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

or

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$X = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

---

find A^{20}

Sol $A^{20} = X D^{20} X^{-1}$

$= X \begin{bmatrix} 4^{20} & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 2^{20} \end{bmatrix} X^{-1}$

---

* (A.m) Algebraic multiplicity of (λ) = times of λ as eigenvalues
* (G.m) Geometric multipilicity of (λ) = number of eigenvectors corresponding of α

A is diagonilizeable if $\boxed{Am = Gm}$ $\forall$ λ of A.

$\Longrightarrow$