## 2.6 Integrating Factors To obtain Exact equations:

Sometimes we can convert a differential equation that is not exact into an exact by multiplying the equation by a suitable integrating factor.

**Case (1)** $M(x,y) + N(x,y)\,y' = 0$ if $M_y \neq N_x$

Find $h(x) = \dfrac{M_y - N_x}{N}$ or $g(y) = \dfrac{N_x - M_y}{M}$

$\Downarrow$ $\Downarrow$

function in x only / function of y only

$\Downarrow$ $\Downarrow$

Then $\boxed{\mu = e^{\int h(x)\,dx}}$ or $\boxed{\mu = e^{\int g(y)\,dy}}$

**Now** The equation will be a new one:

$\mu M(x,y) + \mu N(x,y)\,y' = 0$ it is then exact.

Ex Solve: $(3xy+y^2)dx + (x^2+xy)dy = 0$

Sol: ① First $3xy+y^2 + (x^2+xy)\frac{dy}{dx} = 0$

$$\underbrace{3xy+y^2}_{M} + \underbrace{(x^2+xy)}_{N}\, y' = 0$$

② $\left.\begin{array}{l} M_y = 3x+2y \\ N_x = 2x+y \end{array}\right\} \Rightarrow M_y \neq N_x$ so it is not exact.

③ $h(x) = \frac{M_y - N_x}{N} = \frac{(3x+2y)-(2x+y)}{x^2+xy} = \frac{\cancel{x+y}}{x\cancel{(x+y)}} = \frac{1}{x}$

$h(x) = \frac{1}{x}$

④ $\mu = e^{\int \frac{1}{x}} = e^{\ln x} = x$

⑤ The new exact equation is

$$x(3xy+y^2) + x(x^2+xy)y' = 0$$

$$\underbrace{(3x^2y+xy^2)} + \underbrace{(x^3+x^2y)}\, y' = 0$$

$\left.\begin{array}{l} M_y = 3x^2 + 2xy \\ N_x = 3x^2 + 2xy \end{array}\right\}$ exact.

Solve this exact equation

$$\psi = \int (3x^2 y + xy^2)\, dx = \boxed{x^3 y + \frac{x^2 y^2}{2} + h(y)}$$

$$\psi_y = x^3 + x^2 y + h'(y) = x^3 + x^2 y$$

$\Rightarrow h'(y) = 0$

$\Rightarrow h(y) = c$

The solution is $x^3 y + \frac{x^2 y^2}{2} + c = 0$

$$\equiv \boxed{x^3 y + \frac{x^2 y^2}{2} = C_*}$$

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Q Use the integrating factor $\mu(x,y) = x e^x$ to solve

$(x+2) \sin y \, dx + x \cos y \, dy = 0$

Sol $((x+2) \sin y) + (x \cos y)\, y' = 0$

$M_y = (x+2) \cos y$
$N_x = \cos y$ $\Big\} \neq$

Now $x e^x (x+2) \sin y + x e^x (x \cos y)\, y' = 0$ is exact

$\Downarrow$

$(x^2 e^x + 2x e^x) \sin y + (x^2 e^x \cos y)\, y' = 0$

$\bar{M}_y = (x^2 e^x + 2x e^x) \cos y$
$\bar{N}_x = (x^2 e^x + 2x e^x) \cos y$

$\Downarrow$

Now $\psi = \int (x^2 e^x + 2x e^x)\, dx \; \ldots$

$$\psi = \int (x^2 e^x \cos y)\, dy = \boxed{x^2 e^x \sin y + h(x)}$$

$$\psi_x = (x^2 e^x + 2x e^x) \sin y + h'(x) = x^2 e^x + 2x e^x \sin y$$

$\Rightarrow \quad h'(x) = 0 \quad \Longrightarrow h(x) = c$

and

The solution is: $\quad x^2 e^x \sin y + h(x) = 0$

$$x^2 e^x \sin y + C = 0$$

$$\equiv \boxed{x^2 e^x \sin y = C_1}$$

Example  Solve $\quad y\,dx + (2x - ye^y)\,dy = 0$

solution

$$y + (2x - ye^y)\,y' = 0$$

$$\left.\begin{aligned} M_y &= 1 \\ N_x &= 2 \end{aligned}\right\} \neq \Longrightarrow \text{Not exact}$$

find I.F: $\quad g(y) = \dfrac{N_x - N_y}{M} = \dfrac{2-1}{y} = \dfrac{1}{y}$

Thus $\quad \mu_{(y)} = e^{\int g(y)\,dy} = e^{\int \frac{1}{y}\,dy} = e^{\ln y} = y$

So $\quad \underbrace{y^2}_{} + \underbrace{(2xy - y^2 e^y)}_{} y' = 0 \quad$ is exact!

$$\left.\begin{aligned} \bar{M}_y &= 2y \\ N_x &= 2y \end{aligned}\right\} \Longrightarrow \checkmark$$

$$\Psi = \int y^2\,dx = \boxed{y^2 x + h(y)}$$

$$\Psi_y = \cancel{2y}x + h'(y) = \cancel{2xy} - y^2 e^y$$

$$\Rightarrow h'(y) = y^2 e^y \Longrightarrow h(y) = \int y^2 e^y\,dy$$

$$h(y) = \int y^2 e^y \, dy$$

$$= y^2 e^y - 2y e^y + 2e^y + C$$

| | D. | I. |
|---|---|---|
| + | $y^2$ | $e^y$ |
| − | $2y$ | $e^y$ |
| + | $2$ | $e^y$ |
| | $0$ | $e^y$ |

The solution is $\boxed{y^2 x + y^2 e^y - 2y e^y + 2e^y = C}$

**Q18** page 95 — Show that any separable equation of the form:

$$M(x) + N(y)\, y' = 0$$

is exact.

Page 95

**Pf**

$$\left.\begin{array}{l} M_y = 0 \\ N_x = 0 \end{array}\right\} \Rightarrow \text{exact.}$$

**Note** Any separable equation is exact.

$$\boxed{h(x)}\,dx = \boxed{g(y)}\,dy \qquad \text{separable}$$

$$\equiv \underbrace{h(x)}_{M} - \underbrace{g(y)}_{N}\, y' = 0$$

$$M_y = 0, \quad N_x = 0 \Rightarrow \text{exact.}$$

$u \neq N_x \Rightarrow$ the equation is not exact.

ec. 2.4

## Differences Between Linear and NonLinear Equations:

Existence and Uniqueness of Solutions

[1] For Linear equation : $\boxed{y' + p(t)y = g(t)}$ , $y(t_0) = y_0$

Thm: If $p(t), g(t)$ are continuous on $\alpha < t < \beta$ containing the point $t_0$, then there exists a unique function ~~$y = \phi(t)$~~ that (solution) satisfies the differential equation: $y' + p(t)y = g(t)$ ~~for each t in I~~, and that also satisfies the initial condition $y(t_0) = y_0$

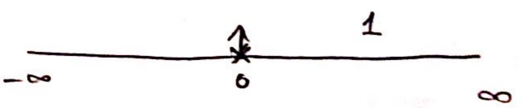
ex find an interval on which the I.V.P

$t y' + 2y = 4t^2$ , $\boxed{y(1) = 2}$ (Largest interval)

has a unique solution

sol $\boxed{y' + \frac{2}{t} y = 4t}$

$p(t) = \frac{2}{t} \Rightarrow$ con on $\mathbb{R} - \{0\}$




$g(t) = 4t \Rightarrow$ con on $\mathbb{R}$

$\Rightarrow p(t)$ and $g(t)$ are cont. on $(-\infty, 0) \cup (0, \infty)$

but $1 \in (0, \infty)$

$\Rightarrow$ The equation has unique solution on $(0, \infty)$

ex ..... $y(-2) = 3$

------ on $(-\infty, 0)$

Find the Largest interval where the solution of

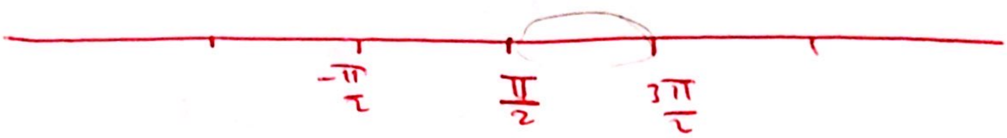
$y' + \tan(t)\, y = \sin t$ , $y(\pi) = 0$ exists.

Sol

$\tan t$ cont. on $\mathbb{R} - \{\mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \dots\}$

$\sin t$ cont. on $\mathbb{R}$

1



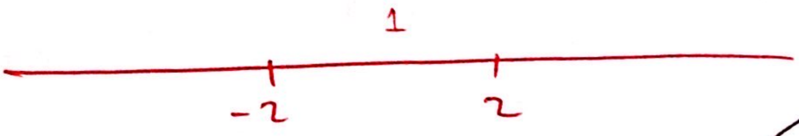

$\pi \in (\frac{\pi}{2}, \frac{3\pi}{2})$

Thus the interval is $(\frac{\pi}{2}, \frac{3\pi}{2})$

Ex $y' + \frac{2}{t^2-4}\, y = 4t^2$ $y(1) = -3$

$\frac{2}{t^2-4}$ is cont. on $\mathbb{R} - \{2, -2\}$

$4t^2$ is cont. on $\mathbb{R}$




$(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$

$1 \in (-2, 2)$

Thus the interval is $(-2, 2)$