CHAPTER
3

Second Order Linear Equations

Linear equations of second order are of crucial importance in the study of differential equations for two main reasons. The first is that linear equations have a rich theoretical structure that underlies a number of systematic methods of solution. Further, a substantial portion of this structure and these methods are understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this chapter for second order equations. Another reason to study second order linear equations is that they are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second order linear differential equations. As an example, we discuss the oscillations of some basic mechanical and electrical systems at the end of the chapter.

omogeneous Equations with Constant Coefficients

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right), \tag{1}$$

where f is some given function. Usually, we will denote the independent variable by t since time is often the independent variable in physical problems, but sometimes we will use x instead. We will use y, or occasionally some other letter, to designate the dependent variable. Equation (1) is said to be **linear** if the function f has the form

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y, \tag{2}$$

that is, if f is linear in y and y'. In Eq. (2) g, p, and q are specified functions of the independent variable t but do not depend on y. In this case we usually rewrite Eq. (1) as

$$y'' + p(t)y' + q(t)y = g(t), \tag{3}$$

where the primes denote differentiation with respect to t. Instead of Eq. (3), we often see the equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t). \tag{4}$$

Of course, if $P(t) \neq 0$, we can divide Eq. (4) by $P(t)$ and thereby obtain Eq. (3) with

$$p(t) = \frac{Q(t)}{P(t)}, \qquad q(t) = \frac{R(t)}{P(t)}, \qquad g(t) = \frac{G(t)}{P(t)}. \tag{5}$$

In discussing Eq. (3) and in trying to solve it, we will restrict ourselves to intervals in which p, q, and g are continuous functions.[1]

If Eq. (1) is not of the form (3) or (4), then it is called **nonlinear**. Analytical investigations of nonlinear equations are relatively difficult, so we will have little to say about them in this book. Numerical or geometical approaches are often more appropriate, and these are discussed in Chapters 8 and 9. In addition, there are two special types of second order nonlinear equations that can be solved by a change of variables that reduces them to first order equations. This procedure is outlined in Problems 28 through 43.

An initial value problem consists of a differential equation such as Eq. (1), (3), or (4) together with a pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{6}$$

where y_0 and y_0' are given numbers. Observe that the initial conditions for a second order equation prescribe not only a particular point (t_0, y_0) through which the graph of the solution must pass, but also the slope y_0' of the graph at that point. It is reasonable to expect that two initial conditions are needed for a second order equation because, roughly speaking, two integrations are required to find a solution and each integration introduces an arbitrary constant. Presumably, two initial conditions will suffice to determine values for these two constants.

A second order linear equation is said to be **homogeneous** if the term $g(t)$ in Eq. (3), or the term $G(t)$ in Eq. (4), is zero for all t. Otherwise, the equation is called **nonhomogeneous**. As a result, the term $g(t)$, or $G(t)$, is sometimes called the nonhomogeneous term. We begin our discussion with homogeneous equations, which we will write in the form

$$P(t)y'' + Q(t)y' + R(t)y = 0. \tag{7}$$

[1]There is a corresponding treatment of higher order linear equations in Chapter 4. If you wish, you may read the appropriate parts of Chapter 4 in parallel with Chapter 3.

Later, in Sections 3.6 and 3.7, we will show that once the homogeneous equation has been solved, it is always possible to solve the corresponding nonhomogeneous equation (4), or at least to express the solution in terms of an integral. Thus the problem of solving the homogeneous equation is the more fundamental one.

In this chapter we will concentrate our attention on equations in which the functions P, Q, and R are constants. In this case, Eq. (7) becomes

$$\boxed{ay'' + by' + cy = 0,} \tag{8}$$

where a, b, and c are given constants. It turns out that Eq. (8) can always be solved easily in terms of the elementary functions of calculus. On the other hand, it is usually much more difficult to solve Eq. (7) if the coefficients are not constants, and a treatment of that case is deferred until Chapter 5.

Before taking up Eq. (8), let us first gain some experience by looking at a simple, but typical, example. Consider the equation

$$y'' - y = 0, \tag{9}$$

which is just Eq. (8) with $a = 1$, $b = 0$, and $c = -1$. In words, Eq. (9) says that we seek a function with the property that the second derivative of the function is the same as the function itself. A little thought will probably produce at least one well-known function from calculus with this property, namely, $y_1(t) = e^t$, the exponential function. A little more thought may also produce a second function, $y_2(t) = e^{-t}$. Some further experimentation reveals that constant multiples of these two solutions are also solutions. For example, the functions $2e^t$ and $5e^{-t}$ also satisfy Eq. (9), as you can verify by calculating their second derivatives. In the same way, the functions $c_1y_1(t) = c_1e^t$ and $c_2y_2(t) = c_2e^{-t}$ satisfy the differential equation (9) for all values of the constants c_1 and c_2. Next, it is of paramount importance to notice that any sum of solutions of Eq. (9) is also a solution. In particular, since $c_1y_1(t)$ and $c_2y_2(t)$ are solutions of Eq. (9), so is the function

$$y = c_1y_1(t) + c_2y_2(t) = c_1e^t + c_2e^{-t} \tag{10}$$

for any values of c_1 and c_2. Again, this can be verified by calculating the second derivative y'' from Eq. (10). We have $y' = c_1e^t - c_2e^{-t}$ and $y'' = c_1e^t + c_2e^{-t}$; thus y'' is the same as y, and Eq. (9) is satisfied.

Let us summarize what we have done so far in this example. Once we notice that the functions $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions of Eq. (9), it follows that the general linear combination (10) of these functions is also a solution. Since the coefficients c_1 and c_2 in Eq. (10) are arbitrary, this expression represents a doubly infinite family of solutions of the differential equation (9).

It is now possible to consider how to pick out a particular member of this infinite family of solutions that also satisfies a given set of initial conditions. For example, suppose that we want the solution of Eq. (9) that also satisfies the initial conditions

$$y(0) = 2, \qquad y'(0) = -1. \tag{11}$$

In other words, we seek the solution that passes through the point (0, 2) and at that point has the slope -1. First, we set $t = 0$ and $y = 2$ in Eq. (10); this gives the equation

$$c_1 + c_2 = 2. \tag{12}$$

Chapter 3:

Section 3.1 : Homogeneous Equations with Constant coefficients :

$$\boxed{a y'' + b y' + c y = 0}$$

To solve this form of O.D.E:

1. Write the characteristic equation

$$\boxed{a r^2 + b r + c = 0}$$

2. find r :

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

3. We have three cases :

real roots

$r_1 \neq r_2$ reals

$y_1 = e^{r_1 t}$

$y_2 = e^{r_2 t}$

G.S. :

$y = c_1 y_1 + c_2 y_2$

real roots

$r_1 = r_2 = r$ real

$y_1 = e^{rt}$

$y_2 = t e^{rt}$

G.S.:

$y = c_1 y_1 + c_2 y_2$

complex roots

$r_1 = \overline{r_2}$ (complex conjucate

$r_1 = \lambda + \mu i$

$r_2 = \lambda - \mu i$

$y_1 = e^{\lambda t} \cos \mu t$

$y_2 = e^{\lambda t} \sin \mu t$

G.S.:

$y = c_1 y_1 + c_2 y_2$

Ex 1 Find the general solution of the O.D.E :

$$y'' + 5y' + 6y = 0$$

Sol

$$r^2 + 5r + 6 = 0$$

$$(r+3)(r+2) = 0$$

$$r = -3, \quad r = -2$$

$$y_1 = e^{-3t}$$

$$y_2 = e^{-2t}$$

G.S. : $y = c_1 e^{-3t} + c_2 e^{-2t}$

H.W $\quad 4y'' - 8y' + 3y = 0$

Ex 2 $\quad y'' + 4y' + 4y = 0$

Sol

$$r^2 + 4r + 4 = 0$$

$$(r+2)(r+2) = 0$$

$$r = -2$$

$$y_1 = e^{-2t}$$

$$y_2 = t e^{-2t}$$

G.S. : $y = c_1 e^{-2t} + c_2 t e^{-2t}$

H.W: $\quad 4y'' + 4y' + y = 0$

Ex 3 Solve $y'' + y' + y = 0$

Sol $r^2 + r + 1 = 0$

$$r = \frac{-1 \mp \sqrt{1-4(1)(1)}}{2(1)} = \frac{-1 \mp \sqrt{-3}}{2} = -\frac{1}{2} \mp \frac{\sqrt{3}}{2} i$$

$$\lambda \qquad \mu$$

$$y_1 = e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t$$

$$y_2 = e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$$

G.S: $y = c_1 e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + c_2 e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$

$= \dots$

Ex 4 Solve: $y'' + 9y = 0$

Sol $r^2 + 9 = 0$

$r = \mp \sqrt{-9}$

$= \mp 3i$

$r = 0 \mp 3i$

$\lambda \quad \mu$

$$y_1 = e^{0t} \cos 3t = \cos 3t$$

$$y_2 = e^{0t} \sin 3t = \sin 3t$$

G.S: $y = c_1 \cos 3t + c_2 \sin 3t$.

H.W $y'' + y = 0$

Ex solve the I.V.P

$y'' - y = 0 \quad , \quad y(0) = 2 \, , \; y'(0) = -1$

Sol $\quad r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = \mp 1$

$y_1 = e^{t}$

$y_2 = e^{-t}$

$$\boxed{y = c_1 e^{t} + c_2 e^{-t}}$$

$y(0) = c_1 e^{0} + c_2 e^{-0}$

$$\boxed{2 = c_1 + c_2} \longrightarrow ①$$

$y' = c_1 e^{t} - c_2 e^{-t}$

$y'(0) = c_1 e^{0} - c_2 e^{-0}$

$$\boxed{-1 = c_1 - c_2} \longrightarrow ②$$

solving ① and ②

$c_1 + c_2 = 2$

$c_1 - c_2 = -1$

$2c_1 = 1 \Rightarrow \boxed{c_1 = \frac{1}{2}}$

$c_2 = 2 - \frac{1}{2} = \boxed{\frac{3}{2} = c_2}$

The solution is

$$\boxed{y = \frac{1}{2} e^{t} + \frac{3}{2} e^{-t}}$$

① It satisfies the eq. $y'' - y = 0$

② " " $y(0) = 2$

③ " " $y'(0) = -1$

H.W Find the solution of the initial value problem

$$4y'' - 8y' + 3y = 0 \quad , \; y(0) = 2$$
$$y'(0) = \frac{1}{2}$$

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$$\boxed{y = -\frac{1}{2} e^{\frac{3t}{2}} + \frac{5}{2} e^{\frac{t}{2}}}$$

1.36

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+ 3.4
3.5

Ex Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0 \quad , \; y(0) = -2$$
$$y'(0) = 1$$

Sol

$$16r^2 - 8r + 145 = 0$$

$$r = \frac{1}{4} \mp 3i$$

$$\lambda \qquad \mu$$

$$y_1 = e^{\frac{1}{4}t} \cos 3t$$
$$y_2 = e^{\frac{1}{4}t} \sin 3t$$

$$y = c_1 e^{\frac{1}{4}t} \cos 3t + c_2 e^{\frac{1}{4}t} \sin 3t \Rightarrow \boxed{y(0) = c_1 = -2}$$

$$y' = e^{\frac{t}{4}} (\cancel{c_1} \cos 3t + c_2 \sin 3t) \Rightarrow$$

$$y'(t) = e^{\frac{t}{4}} (-3c_1 \sin 3t + 3c_2 \cos 3t) + \frac{1}{4} e^{\frac{t}{4}} (c_1 \cos 3t + c_2 \sin 3t)$$

$$y'(0) = 3c_2 + \frac{1}{4} c_1 = 1 \Rightarrow \boxed{c_2 = \frac{1}{2}}$$

$$\boxed{y = -2 e^{\frac{1}{4}t} \cos 3t + \frac{1}{2} e^{\frac{1}{4}t} \sin 3t}$$

(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.
(c) Determine the smallest value of β for which $y_m \geq 4$.
(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

27. Find an equation of the form $ay'' + by' + cy = 0$ for which all solutions approach a multiple of e^{-t} as $t \to \infty$.

Equations with the Dependent Variable Missing. For a second order differential equation of the form $y'' = f(t, y')$, the substitution $v = y'$, $v' = y''$ leads to a first order equation of the form $v' = f(t, v)$. If this equation can be solved for v, then y can be obtained by integrating $dy/dt = v$. Note that one arbitrary constant is obtained in solving the first order equation for v, and a second is introduced in the integration for y. In each of Problems 28 through 33 use this substitution to solve the given equation.

28. $t^2y'' + 2ty' - 1 = 0, \quad t > 0$
29. $ty'' + y' = 1, \quad t > 0$
30. $y'' + t(y')^2 = 0$
31. $2t^2y'' + (y')^3 = 2ty', \quad t > 0$
32. $y'' + y' = e^{-t}$
33. $t^2y'' = (y')^2, \quad t > 0$

Equations with the Independent Variable Missing. If a second order differential equation has the form $y'' = f(y, y')$, then the independent variable t does not appear explicitly, but only through the dependent variable y. If we let $v = y'$, then we obtain $dv/dt = f(y, v)$. Since the right side of this equation depends on y and v, rather than on t and v, this equation is not of the form of the first order equations discussed in Chapter 2. However, if we think of y as the independent variable, then by the chain rule $dv/dt = (dv/dy)(dy/dt) = v(dv/dy)$. Hence the original differential equation can be written as $v(dv/dy) = f(y, v)$. Provided that this first order equation can be solved, we obtain v as a function of y. A relation between y and t results from solving $dy/dt = v(y)$. Again, there are two arbitrary constants in the final result. In each of Problems 34 through 39 use this method to solve the given differential equation.

34. $yy'' + (y')^2 = 0$
35. $y'' + y = 0$
36. $y'' + y(y')^3 = 0$
37. $2y^2y'' + 2y(y')^2 = 1$
38. $yy'' - (y')^3 = 0$
39. $y'' + (y')^2 = 2e^{-y}$

Hint: In Problem 39 the transformed equation is a Bernoulli equation. See Problem 27 in Section 2.4.

In each of Problems 40 through 43 solve the given initial value problem using the methods of Problems 28 through 39.

40. $y'y'' = 2, \quad y(0) = 1, \quad y'(0) = 2$
41. $y'' - 3y^2 = 0, \quad y(0) = 2, \quad y'(0) = 4$
42. $(1 + t^2)y'' + 2ty' + 3t^{-2} = 0, \quad y(1) = 2, \quad y'(1) = -1$
43. $y'y'' - t = 0, \quad y(1) = 2, \quad y'(1) = 1$

3.2 Fundamental Solutions of Linear Homogeneous Equations

In the preceding section we showed how to solve some differential equations of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. Now we build on those results to provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations. In turn, this understanding will assist us in finding the solutions of other problems that we will encounter later.

In developing the theory of linear differential equations, it is helpful to introduce a differential operator notation. Let p and q be continuous functions on an open interval I, that is, for $\alpha < t < \beta$. The cases $\alpha = -\infty$, or $\beta = \infty$, or both, are included. Then, for any function ϕ that is twice differentiable on I, we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{1}$$

Note that $L[\phi]$ is a function on I. The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

For example, if $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin 3t$, then

$$\begin{aligned} L[\phi](t) &= (\sin 3t)'' + t^2(\sin 3t)' + (1 + t)\sin 3t \\ &= -9\sin 3t + 3t^2\cos 3t + (1 + t)\sin 3t. \end{aligned}$$

The operator L is often written as $L = D^2 + pD + q$, where D is the derivative operator.

In this section we study the second order linear homogeneous equation $L[\phi](t) = 0$. Since it is customary to use the symbol y to denote $\phi(t)$, we will usually write this equation in the form

$$L[y] = y'' + p(t)y' + q(t)y = 0. \tag{2}$$

With Eq. (2) we associate a set of initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{3}$$

where t_0 is any point in the interval I, and y_0 and y_0' are given real numbers. We would like to know whether the initial value problem (2), (3) always has a solution, and whether it may have more than one solution. We would also like to know whether anything can be said about the form and structure of solutions that might be helpful in finding solutions of particular problems. Answers to these questions are contained in the theorems in this section.

The fundamental theoretical result for initial value problems for second order linear equations is stated in Theorem 3.2.1, which is analogous to Theorem 2.4.1 for first order linear equations. The result applies equally well to nonhomogeneous equations, so the theorem is stated in that form.

Theorem 3.2.1 Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{4}$$

where p, q, and g are continuous on an open interval I. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

We emphasize that the theorem says three things:

1. The initial value problem *has* a solution; in other words, a solution *exists*.
2. The initial value problem has *only one* solution; that is, the solution is *unique*.

3. The solution ϕ is defined *throughout the interval I* where the coefficients are continuous and is at least twice differentiable there.

For some problems some of these assertions are easy to prove. For example, we found in Section 3.1 that the initial value problem

$$y'' - y = 0, \qquad y(0) = 2, \quad y'(0) = -1 \tag{5}$$

has the solution

$$y = \tfrac{1}{2}e^{t} + \tfrac{3}{2}e^{-t}. \tag{6}$$

The fact that we found a solution certainly establishes that a solution exists for this initial value problem. Further, the solution (6) is twice differentiable, indeed differentiable any number of times, throughout the interval $(-\infty, \infty)$ where the coefficients in the differential equation are continuous. On the other hand, it is not obvious, and is more difficult to show, that the initial value problem (5) has no solutions other than the one given by Eq. (6). Nevertheless, Theorem 3.2.1 states that this solution is indeed the only solution of the initial value problem (5).

However, for most problems of the form (4), it is not possible to write down a useful expression for the solution. This is a major difference between first order and second order linear equations. Therefore, all parts of the theorem must be proved by general methods that do not involve having such an expression. The proof of Theorem 3.2.1 is fairly difficult, and we do not discuss it here.[2] We will, however, accept Theorem 3.2.1 as true and make use of it whenever necessary.

EXAMPLE 1

Find the longest interval in which the solution of the initial value problem

$$\boxed{(t^2 - 3t)y'' + ty' - (t + 3)y = 0,} \qquad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

If the given differential equation is written in the form of Eq. (4), then $p(t) = 1/(t-3)$, $q(t) = -(t+3)/t(t-3)$, and $g(t) = 0$. The only points of discontinuity of the coefficients are $t = 0$ and $t = 3$. Therefore, the longest open interval, containing the initial point $t = 1$, in which all the coefficients are continuous is $0 < t < 3$. Thus, this is the longest interval in which Theorem 3.2.1 guarantees that the solution exists.

$y'' + \frac{t\,y'}{t^2-3t} + \frac{t+3}{t^2-3t}\,y = 0$ 0 3

EXAMPLE 2

Find the unique solution of the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where p and q are continuous in an open interval I containing t_0.

The function $y = \phi(t) = 0$ for all t in I certainly satisfies the differential equation and initial conditions. By the uniqueness part of Theorem 3.2.1 it is the only solution of the given problem.

[2]A proof of Theorem 3.2.1 may be found, for example, in Chapter 6, Section 8 of the book by Coddington listed in the references.

$y'' + \frac{\cancel{t}\ 1}{\cancel{t}(t-3)}\,y' + \frac{t+3}{t(t-3)}\,y = 0$

0 3

Let us now assume that y_1 and y_2 are two solutions of Eq. (2); in other words,

$$L[y_1] = y_1'' + py_1' + qy_1 = 0, \tag{7}$$

and similarly for y_2. Then, just as in the examples in Section 3.1, we can generate more solutions by forming linear combinations of y_1 and y_2. We state this result as a theorem.

Theorem 3.2.2 **(Principle of Superposition)** If y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

A special case of Theorem 3.2.2 occurs if either c_1 or c_2 is zero. Then we conclude that any multiple of a solution of Eq. (2) is also a solution.

To prove Theorem 3.2.2 we need only substitute

$$y = c_1y_1(t) + c_2y_2(t) \tag{8}$$

for y in Eq. (2). The result is

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= [c_1y_1 + c_2y_2]'' + p[c_1y_1 + c_2y_2]' + q[c_1y_1 + c_2y_2] \\ &= c_1y_1'' + c_2y_2'' + c_1py_1' + c_2py_2' + c_1qy_1 + c_2qy_2 \\ &= c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2] \\ &= c_1L[y_1] + c_2L[y_2]. \end{aligned}$$

Since $L[y_1] = 0$ and $L[y_2] = 0$, it follows that $L[c_1y_1 + c_2y_2] = 0$ also. Therefore, regardless of the values of c_1 and c_2, y as given by Eq. (8) does satisfy the differential equation (2) and the proof of Theorem 3.2.2 is complete.

Theorem 3.2.2 states that, beginning with only two solutions of Eq. (2), we can construct a doubly infinite family of solutions by means of Eq. (8). The next question is whether all solutions of Eq. (2) are included in Eq. (8), or whether there may be other solutions of a different form. We begin to address this question by examining whether the constants c_1 and c_2 in Eq. (8) can be chosen so as to satisfy the initial conditions (3). These initial conditions require c_1 and c_2 to satisfy the equations

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0'. \end{aligned} \tag{9}$$

Upon solving Eqs. (9) for c_1 and c_2, we find that

$$c_1 = \frac{y_0y_2'(t_0) - y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \qquad c_2 = \frac{-y_0y_1'(t_0) + y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \tag{10}$$

or, in terms of determinants,

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}. \tag{11}$$

With these values for c_1 and c_2 the expression (8) satisfies the initial conditions (3) as well as the differential equation (2).

In order for the expressions for c_1 and c_2 in Eqs. (10) or (11) to make sense, it is necessary that the denominators be nonzero. For both c_1 and c_2 the denominator is the same, namely, the determinant

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0). \tag{12}$$

The determinant W is called the **Wronskian**[3] **determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2. Sometimes we use the more extended notation $W(y_1, y_2)(t_0)$ to stand for the expression on the right side of Eq. (12), thereby emphasizing that the Wronskian depends on the functions y_1 and y_2, and that it is evaluated at the point t_0. The preceding argument suffices to establish the following result.

Theorem 3.2.3 Suppose that y_1 and y_2 are two solutions of Eq. (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the Wronskian

$$W = y_1y_2' - y_1'y_2 \quad = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is not zero at the point t_0 where the initial conditions (3),

$$y(t_0) = y_0, \qquad y'(t_0) = y_0',$$

are assigned. Then there is a choice of the constants c_1, c_2 for which $y = c_1y_1(t) + c_2y_2(t)$ satisfies the differential equation (2) and the initial conditions (3).

EXAMPLE 3

In Example 1 of Section 3.1 we found that $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions of the differential equation

$$\boxed{y'' + 5y' + 6y = 0.}$$

Find the Wronskian of y_1 and y_2.

The Wronskian of these two functions is

$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}.$$

[3] Wronskian determinants are named for Jósef Maria Hoëné-Wronski (1776–1853), who was born in Poland but spent most of his life in France. Wronski was a gifted but troubled man, and his life was marked by frequent heated disputes with other individuals and institutions.

Since W is nonzero for all values of t, the functions y_1 and y_2 can be used to construct solutions of the given differential equation, together with initial conditions prescribed at any value of t. One such initial value problem was solved in Example 2 of Section 3.1.

The next theorem justifies the term "general solution" that we introduced in Section 3.1 for the linear combination $c_1y_1 + c_2y_2$.

Theorem 3.2.4 If y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and if there is a point t_0 where the Wronskian of y_1 and y_2 is nonzero, then the family of solutions

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of Eq. (2).

Let ϕ be any solution of Eq. (2). To prove the theorem we must show that ϕ is included in the linear combination $c_1y_1 + c_2y_2$; that is, for some choice of the constants c_1 and c_2, the linear combination is equal to ϕ. Let t_0 be a point where the Wronskian of y_1 and y_2 is nonzero. Then evaluate ϕ and ϕ' at this point and call these values y_0 and y_0', respectively; thus

$$y_0 = \phi(t_0), \qquad y_0' = \phi'(t_0).$$

Next, consider the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{13}$$

The function ϕ is certainly a solution of this initial value problem. On the other hand, since $W(y_1, y_2)(t_0)$ is nonzero, it is possible (by Theorem 3.2.3) to choose c_1 and c_2 so that $y = c_1y_1(t) + c_2y_2(t)$ is also a solution of the initial value problem (13). In fact, the proper values of c_1 and c_2 are given by Eqs. (10) or (11). The uniqueness part of Theorem 3.2.1 guarantees that these two solutions of the same initial value problem are actually the same function; thus, for the proper choice of c_1 and c_2,

$$\phi(t) = c_1y_1(t) + c_2y_2(t),$$

and therefore ϕ is included in the family of functions of $c_1y_1 + c_2y_2$. Finally, since ϕ is an *arbitrary* solution of Eq. (2), it follows that *every* solution of this equation is included in this family. This completes the proof of Theorem 3.2.4.

Theorem 3.2.4 states that, as long as the Wronskian of y_1 and y_2 is not everywhere zero, the linear combination $c_1y_1 + c_2y_2$ contains all solutions of Eq. (2). It is therefore natural (and we have already done this in the preceding section) to call the expression

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary constant coefficients the **general solution** of Eq. (2). The solutions y_1 and y_2, with a nonzero Wronskian, are said to form a **fundamental set of solutions** of Eq. (2).

To restate the result of Theorem 3.2.4 in slightly different language: To find the general solution, and therefore all solutions, of an equation of the form (2), we need only find two solutions of the given equation whose Wronskian is nonzero. We did precisely this in several examples in Section 3.1, although there we did not calculate the Wronskians. You should now go back and do that, thereby verifying that all the solutions we called "general solutions" in Section 3.1 do satisfy the necessary Wronskian condition. Alternatively, the following example includes all those mentioned in Section 3.1, as well as many other problems of a similar type.

EXAMPLE 4

Suppose that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of an equation of the form (2). Show that they form a fundamental set of solutions if $r_1 \neq r_2$.

We calculate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)\exp[(r_1 + r_2)t].$$

Since the exponential function is never zero, and since $r_2 - r_1 \neq 0$ by the statement of the problem, it follows that W is nonzero for every value of t. Consequently, y_1 and y_2 form a fundamental set of solutions.

EXAMPLE 5

Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \qquad t > 0. \tag{14}$$

We will show in Section 5.5 how to solve Eq. (14); see also Problem 38 in Section 3.4. However, at this stage we can verify by direct substitution that y_1 and y_2 are solutions of the differential equation. Since $y_1'(t) = \frac{1}{2}t^{-1/2}$ and $y_1''(t) = -\frac{1}{4}t^{-3/2}$, we have

$$2t^2(-\tfrac{1}{4}t^{-3/2}) + 3t(\tfrac{1}{2}t^{-1/2}) - t^{1/2} = (-\tfrac{1}{2} + \tfrac{3}{2} - 1)t^{1/2} = 0.$$

Similarly, $y_2'(t) = -t^{-2}$ and $y_2''(t) = 2t^{-3}$, so

$$2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = (4 - 3 - 1)t^{-1} = 0.$$

Next we calculate the Wronskian W of y_1 and y_2:

$$W = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\tfrac{3}{2}t^{-3/2}. \tag{15}$$

Since $W \neq 0$ for $t > 0$, we conclude that y_1 and y_2 form a fundamental set of solutions there.

In several cases, we have been able to find a fundamental set of solutions, and therefore the general solution, of a given differential equation. However, this is often a difficult task, and the question may arise as to whether or not a differential equation of the form (2) always has a fundamental set of solutions. The following theorem provides an affirmative answer to this question.

__Ex:__

Show that $y_1(t) = t^{\frac{1}{2}}$, $y_2(t) = t^{-1}$ form a fundamental set of solutions of $2t^2 y'' + 3t y' - y = 0$, $t > 0$

~~and~~

__Solution__ ① y_1 satisfies the O.D.E :

$$y_1(t) = t^{\frac{1}{2}}.$$
$$y_1'(t) = \tfrac{1}{2} t^{-\frac{1}{2}}$$
$$y_1''(t) = -\tfrac{1}{4} t^{-\frac{3}{2}}$$

$$2t^2\left(-\tfrac{1}{4} t^{-\frac{3}{2}}\right) + 3t\left(\tfrac{1}{2} t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}}$$
$$-\tfrac{1}{2} t^{\frac{1}{2}} + \tfrac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0$$

② $y_2(t) = t^{-1}$
$$y_2'(t) = -t^{-2}$$
$$y_2''(t) = 2t^{-3}$$

$$2t^2(2t^{-3}) + 3t(-t^{-2}) + -t^{-1} = 4t^{-1} + -3t^{-1} - t^{-1} = 0$$

③ $$W(y_1, y_2) = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \tfrac{1}{2} t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \tfrac{1}{2} t^{-\frac{3}{2}} = -\tfrac{3}{2} t^{-\frac{3}{2}}$$

$W(y_1, y_2) \neq 0$ for $t > 0$

Thus y_1, y_2 form a fundamental set of solutions.

In each of Problems 33 through 35 use the result of Problem 32 to find the adjoint of the given differential equation.

33. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel's equation
34. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, Legendre's equation
35. $y'' - xy = 0$, Airy's equation
36. For the second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$, show that the adjoint of the adjoint equation is the original equation.
37. A second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be self-adjoint if its adjoint is the same as the original equation. Show that a necessary condition for this equation to be self-adjoint is that $P'(x) = Q(x)$. Determine whether each of the equations in Problems 33 through 35 is self-adjoint.

3.3 Linear Independence and the Wronskian

The representation of the general solution of a second order linear homogeneous differential equation as a linear combination of two solutions whose Wronskian is not zero is intimately related to the concept of linear independence of two functions. This is a very important algebraic idea and has significance far beyond the present context; we briefly discuss it in this section.

We will refer to the following basic property of systems of linear homogeneous algebraic equations. Consider the two–by–two system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0, \\ a_{21}x_1 + a_{22}x_2 &= 0, \end{aligned} \tag{1}$$

and let $\Delta = a_{11}a_{22} - a_{12}a_{21}$ be the corresponding determinant of coefficients. Then $x = 0$, $y = 0$ is the only solution of the system (1) if and only if $\Delta \neq 0$. Further, the system (1) has nonzero solutions if and only if $\Delta = 0$.

Two functions f and g are said to be **linearly dependent** on an interval I if there exist two constants k_1 and k_2, not both zero, such that

$$k_1 f(t) + k_2 g(t) = 0 \tag{2}$$

for all t in I. The functions f and g are said to be **linearly independent** on an interval I if they are not linearly dependent; that is, Eq. (2) holds for all t in I only if $k_1 = k_2 = 0$. In Section 4.1 these definitions are extended to an arbitrary number of functions. Although it may be difficult to determine whether a large set of functions is linearly independent or linearly dependent, it is usually easy to answer this question for a set of only two functions: they are linearly dependent if they are proportional to each other, and linearly independent otherwise. The following examples illustrate these definitions.

EXAMPLE 1

Determine whether the functions $\sin t$ and $\cos(t - \pi/2)$ are linearly independent or linearly dependent on an arbitrary interval.

... to the two functions $f(t) = e^t$ and $g(t) = e^{2t}$ discussed in ... any point t_0 we have

$$W(f, g)(t_0) = \begin{vmatrix} e^{t_0} & e^{2t_0} \\ e^{t_0} & 2e^{2t_0} \end{vmatrix} = e^{3t_0} \neq 0. \tag{6}$$

The functions e^t and e^{2t} are therefore linearly independent on any interval.

You should be careful not to read too much into Theorem 3.3.1. In particular, two functions f and g may be linearly independent even though $W(f, g)(t) = 0$ for every t in the interval I. This is illustrated in Problem 28.

Now let us examine further the properties of the Wronskian of two solutions of a second order linear homogeneous differential equation. The following theorem, perhaps surprisingly, gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

...m 3.3.2 **(Abel's Theorem)**[4] **If y_1 and y_2 are solutions of the differential equation**

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{7}$$

where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)\, dt\right], \tag{8}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W(y_1, y_2)(t)$ is either zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

To prove Abel's theorem we start by noting that y_1 and y_2 satisfy

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= 0, \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0. \end{aligned} \tag{9}$$

If we multiply the first equation by $-y_2$, the second by y_1, and add the resulting equations, we obtain

$$(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0. \tag{10}$$

Next, we let $W(t) = W(y_1, y_2)(t)$ and observe that

$$W' = y_1 y_2'' - y_1'' y_2. \tag{11}$$

Then we can write Eq. (10) in the form

$$W' + p(t)W = 0. \tag{12}$$

[4] The result in Theorem 3.3.2 was derived by the Norwegian mathematician Niels Henrik Abel (1802–1829) in 1827 and is known as Abel's formula. Abel also showed that there is no general formula for solving a quintic, or fifth degree, polynomial equation in terms of explicit algebraic operations on the coefficients, thereby resolving a question that had been open since the sixteenth century. His greatest contributions, however, were in analysis, particularly in the study of elliptic functions. Unfortunately, his work was not widely noticed until after his death. The distinguished French mathematician Legendre called it a "monument more lasting than bronze."

Equation (12) can be solved immediately since it is both a first order linear equation (Section 2.1) and a separable equation (Section 2.2). Thus

$$W(t) = c \exp\left[-\int p(t)\, dt\right], \tag{13}$$

where c is a constant. The value of c depends on which pair of solutions of Eq. (7) is involved. However, since the exponential function is never zero, $W(t)$ is not zero unless $c = 0$, in which case $W(t)$ is zero for all t, which completes the proof of Theorem 3.3.2.

Note that the Wronskians of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant, and that the Wronskian of any fundamental set of solutions can be determined, up to a multiplicative constant, without solving the differential equation.

EXAMPLE 3

In Example 5 of Section 3.2 we verified that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions of the equation

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0. \tag{14}$$

Verify that the Wronskian of y_1 and y_2 is given by Eq. (13).

From the example just cited we know that $W(y_1, y_2)(t) = -(3/2)t^{-3/2}$. To use Eq. (13) we must write the differential equation (14) in the standard form with the coefficient of y'' equal to 1. Thus we obtain

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,$$

so $p(t) = 3/2t$. Hence

$$\begin{aligned} W(y_1, y_2)(t) &= c \exp\left[-\int \frac{3}{2t}\, dt\right] = c \exp\left(-\frac{3}{2}\ln t\right) \\ &= c\, t^{-3/2}. \end{aligned} \tag{15}$$

Equation (15) gives the Wronskian of any pair of solutions of Eq. (14). For the particular solutions given in this example we must choose $c = -3/2$.

A stronger version of Theorem 3.3.1 can be established if the two functions involved are solutions of a second order linear homogeneous differential equation.

Theorem 3.3.3 Let y_1 and y_2 be the solutions of Eq. (7),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I. Then y_1 and y_2 are linearly dependent on I if and only if $W(y_1, y_2)(t)$ is zero for all t in I. Alternatively, y_1 and y_2 are linearly independent on I if and only if $W(y_1, y_2)(t)$ is never zero in I.

Of course, we know by Theorem 3.3.2 that $W(y_1, y_2)(t)$ is either everywhere zero or nowhere zero in I. In proving Theorem 3.3.3, observe first that if y_1 and y_2 are linearly dependent, then $W(y_1, y_2)(t)$ is zero for all t in I by Theorem 3.3.1. It remains to prove the converse; that is, if $W(y_1, y_2)(t)$ is zero throughout I, then y_1 and y_2 are linearly dependent. Let t_0 be any point in I; then necessarily $W(y_1, y_2)(t_0) = 0$. Consequently, the system of equations

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= 0, \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= 0 \end{aligned} \tag{16}$$

for c_1 and c_2 has a nontrivial solution. Using these values of c_1 and c_2, let $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$. Then ϕ is a solution of Eq. (7), and by Eqs. (16) ϕ also satisfies the initial conditions

$$\phi(t_0) = 0, \qquad \phi'(t_0) = 0. \tag{17}$$

Therefore, by the uniqueness part of Theorem 3.2.1, or by Example 2 of Section 3.2, $\phi(t) = 0$ for all t in I. Since $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$ with c_1 and c_2 not both zero, this means that y_1 and y_2 are linearly dependent. The alternative statement of the theorem follows immediately.

We can now summarize the facts about fundamental sets of solutions, Wronskians, and linear independence in the following way. Let y_1 and y_2 be solutions of Eq. (7),

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I. Then the following four statements are equivalent, in the sense that each one implies the other three:

1. The functions y_1 and y_2 are a fundamental set of solutions on I.
2. The functions y_1 and y_2 are linearly independent on I.
3. $W(y_1, y_2)(t_0) \neq 0$ for some t_0 in I.
4. $W(y_1, y_2)(t) \neq 0$ for all t in I.

It is interesting to note the similarity between second order linear homogeneous differential equations and two-dimensional vector algebra. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be linearly dependent if there are two scalars k_1 and k_2, not both zero, such that $k_1\mathbf{a} + k_2\mathbf{b} = \mathbf{0}$; otherwise, they are said to be linearly independent. Let $\mathbf{i}$ and $\mathbf{j}$ be unit vectors directed along the positive x and y axes, respectively. Since $k_1\mathbf{i} + k_2\mathbf{j} = \mathbf{0}$ only if $k_1 = k_2 = 0$, the vectors $\mathbf{i}$ and $\mathbf{j}$ are linearly independent. Further, we know that any vector $\mathbf{a}$ with components a_1 and a_2 can be written as $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, that is, as a linear combination of the two linearly independent vectors $\mathbf{i}$ and $\mathbf{j}$. It is not difficult to show that any vector in two dimensions can be expressed as a linear combination of any two linearly independent two-dimensional vectors (see Problem 14). Such a pair of linearly independent vectors is said to form a basis for the vector space of two-dimensional vectors.

The term **vector space** is also applied to other collections of mathematical objects that obey the same laws of addition and multiplication by scalars that geometric vectors do. For example, it can be shown that the set of functions that are twice differentiable on the open interval I forms a vector space. Similarly, the set V of functions satisfying Eq. (7) also forms a vector space.

Since every member of V can be expressed as a linear combination of two linearly independent members y_1 and y_2, we say that such a pair forms a basis for V. This leads to the conclusion that V is two-dimensional; therefore, it is analogous in many respects to the space of geometric vectors in a plane. Later we find that the set of solutions of an

$$y'' + p(t)y' + q(t)y = 0$$

$$W(y_1, y_2) = c \exp\left(-\int p(t)\,dt\right)$$

where c is a constant depends on y_1, y_2 but not t

Ex For the O.D.E :

$2t^2 y'' + 3ty' - y = 0 \quad , t > 0$ find the Wronskian of y_1 and y_2.

sol

$$y'' + \frac{3t}{2t^2} y' - \frac{1}{2t^2} y = 0$$

$$y'' + \underbrace{\frac{3}{2t}}_{p(t)} y' - \frac{1}{2t^2} y = 0$$

$$W(y_1, y_2) = c \exp\left(-\int \frac{3}{2t}\,dt\right)$$

$$= c \exp\left(-\tfrac{3}{2}\ln t\right)$$

$$= c \exp \ln t^{-\frac{3}{2}}$$

$$= c\, t^{-\frac{3}{2}}$$

① $e^{\ln x} = x$

② $\ln x^r = r \ln x$