e.g: Find the general solution of

$$y'' + y' + y = 0.$$

e.g: Find the general solution of

$$y'' + 9y = 0.$$

e.g: Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0 \qquad y(0) = -2 \qquad y'(0) = 1$$

e.g: Solve the differential equation

$$y'' + 4y' + 4y = 0.$$

e.g: Find the solution of the initial value problem.

$$y'' - y' + 0.25y = 0 \qquad y(0) = 2, \quad y'(0) = \frac{1}{3}$$

Reduction of order

It is not easy to solve D.E with variable coefficient. We have this method to solve such problem, if we get one of the solution say y_1 then the second solution

$$y_2 = v(x)\, y_1(x)$$

$$y_2 = y_1(x) \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\, dx$$

where $\boxed{y'' + p(x)y' + q(x)y = 0}$ (coefficient of $y'' = 1$)

e.g1 Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2 y'' + 3t y' - y = 0 \qquad t > 0.$$

find a second linearly independent solution

soln

$$2t^2 y'' + 3t y' - y = 0.$$

$$y'' + \frac{3}{2t} y' - \frac{1}{2t^2} y = 0.$$

$$y(t) = t^{-1}$$

$$y_2(t) = y_1(t) \int \frac{1}{(y_1(t))^2} e^{-\int \frac{3}{2t} dt} dt$$

$$= t^{-1} \int t^2 e^{-\frac{3}{2}\ln t} dt = t^{-1} \int t^2 t^{-\frac{3}{2}} dt$$

$$= t^{-1} \int t^{\frac{1}{2}} dt = \frac{2}{3} t^{-1} t^{\frac{3}{2}}$$

$$= \frac{2}{3} t^{\frac{1}{2}}$$

$$\boxed{y_2 = \frac{2}{3} t^{\frac{1}{2}}}$$

the general solution

$$y = c_1 t^{-1} + c_2 t^{\frac{1}{2}}$$

e.g 2 If $y_1 = e^t$ is a solution of the ODE : $y'' - \frac{2t+2}{t} y' + \frac{t+2}{t} y = 0$

find the G.S. of the ODE .

Sol

$$y_2 = e^t \int \frac{1}{(e^t)^2} e^{-\int -\frac{2t+2}{t} dt} dt$$

$$= e^t \int e^{-2t} e^{2t + 2\ln t} dt$$

$$= e^t \int e^{-2t} e^{2t} e^{\ln t^2} dt$$

$$= e^t \int t^2 dt = \boxed{e^t \frac{t^3}{3}} \Rightarrow y_2 = t^3 e^t$$

G.S: $y = C_1 e^t + C_2 t^3 e^t$

3.6 Nonhomogeneous Equations ; Method of Undetermined Coefficients :

$$ay'' + by' + cy = g(t) \quad , \quad g(t) \neq 0$$

The general solution is $y = y_p + y_c$

y_c : is the solution of : $ay'' + by' + cy = 0$

y_p : is a particular solution of : $ay'' + by' + cy = g(t)$

The undetermined coefficients method is used to find the particular solution if $g(t)$ is :
exponential, sine, cosine, polynomial or sums or products of these functions.

$g(t)$	y_p
polynomial	$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
$3x$	$Ax + B$
5	A
$2x^2 + 1$	$Ax^2 + Bx + C$
$5x^3 + x$	$Ax^3 + Bx^2 + Cx + D$
αe^{at}	$A e^{at}$
$3e^{2t}$	$A e^{2t}$
$\frac{4}{e^t}$	$A e^{-t}$

[illegible]

$g(t)$	y_p
$\alpha \cos at$	$A \cos at + B \sin at$
$\alpha \sin at$	$A \cos at + B \sin at$
$\alpha \cos at \pm \beta \sin at$	$A \cos at + B \sin at$
$2 \cos 3t$	$A \cos 3t + B \sin 3t$
$-3 \sin 5t$	$A \cos 5t + B \sin 5t$
$2 \cos 3t + 7 \sin 3t$	$A \cos 3t + B \sin 3t$
$p(x)\, e^{ax}$	$(a_n x^n + \cdots + a_0)\, e^{ax}$
$x\, e^{2x}$	$(Ax + B)\, e^{2x}$
$p(x) \sin ax$	$(a_n x^n + \cdots + a_0) \cos ax +$ $(b_n x^n + \cdots + b_0) \sin ax$
$p(x)\, e^{ax} \sin ax$	$e^{ax}(a_n x^n + \cdots + a_0) \cos ax +$ $e^{ax}(b_n x^n + \cdots + b_0) \sin ax$
$x^2 e^{x} \sin 2x$	$e^{x}(Ax^2 + Bx + C) \cos 2x +$ $e^{x}(Dx^2 + Lx + M) \sin 2x$

Note : We will multiply the particular solution by X^s where $s = 0, 1, 2, \ldots$ (take the smallest if ...) to ensure that no term in Y_p is a solution of the Hom. eq.

Ex ① Find a particular solution of $y''-3y'-4y=3e^{2t}$

Sol First find y_c: $y''-3y'-4y=0$

$$r^2-3r-4=0$$

$$(r-4)(r+1)=0 \implies r=4,-1$$

$y_1=e^{4t}$

$y_2=e^{-t}$

$$\boxed{y_c=c_1e^{4t}+c_2e^{-t}}$$

Now find y_p: $y''-3y'-4y=3e^{2t}$

$$y_p=Ae^{2t}$$

To find A: $y_p'=2Ae^{2t}$

$y_p''=4Ae^{2t}$

$$4Ae^{2t}-3(2Ae^{2t})-4Ae^{2t}=3e^{2t}$$

$$A\left[-6e^{2t}\right]=3e^{2t} \implies A=-\frac{1}{2}$$

Thus The particular solution is

$$\boxed{y_p=-\frac{1}{2}e^{2t}}$$

The solution of $y''-3y'-4y=3e^{2t}$ is

$$y=y_p+y_c \implies \boxed{y=-\frac{1}{2}e^{2t}+c_1e^{4t}+c_2e^{-t}}$$

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Ex 2 Find the solution of $y'' - 3y' - 4y = 2\sin t$

sol First: by Ex 1 $y_c = c_1 e^{4t} + c_2 e^{-t}$

Now

$$y_p = A\cos t + B\sin t$$

$$y_p' = -A\sin t + B\cos t$$

نعوضهم في المعادلة الأصلية

$$y_p'' = -A\cos t - B\sin t$$

$$2\sin t = -A\cos t - B\sin t - 3[-A\sin t + B\cos t] - 4[A\cos t + B\sin t]$$

$\Rightarrow$

$$2\sin t = (-B + 3A - 4B)\sin t \Rightarrow \boxed{2 = 3A - 5B} \longrightarrow (1)$$

$$0\cos t = (-A - 3B - 4A)\cos t \Rightarrow \boxed{0 = -5A - 3B} \longrightarrow (2)$$

$$B = -\frac{5}{17}, \quad A = \frac{3}{17}$$

$$\boxed{y_p = \frac{3}{17}\cos t - \frac{5}{17}\sin t}$$

The solution is:

$$\boxed{y = c_1 e^{4t} + c_2 e^{-t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t}$$

Ex (3) Find a particular solution of $y'' - 3y' - 4y = 4t^2 - 1$

Sol **First** $y_c = c_1 e^{4t} + c_2 e^{-t}$

Now $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$$2A - 3(2Ax - B) - 4(Ax^2 + Bx + C) = 4t^2 - 1$$

$: \quad 2A + 3B - 4C = -1 \quad \text{———} \; (1)$

$: \quad -6A - 4B = 0 \quad \text{———} \; (2)$

$: \quad -4A = 4 \quad \text{———} \; (3)$

By (3) : $-4A = 4 \implies \boxed{A = -1}$

By (2) : $-6A - 4B = 0 \implies -6(-1) = 4B \implies \boxed{B = \frac{3}{2}}$

By (1) : $2(-1) + 3(\frac{3}{2}) - 4C = -1$

$-1 + \frac{9}{2} = 4C \implies \boxed{C = \frac{7}{8}}$

$$\boxed{y_p = -x^2 + \frac{3}{2}x + \frac{7}{8}}$$

Ex Find a particular solution of $y'' - 3y' - 4y = -8e^{t}\cos 2t$

Sol

$$y_c = c_1 e^{4t} + c_2 e^{-t}$$

$$y_p = A e^{t}\cos 2t + B e^{t}\sin 2t$$

$$y_p' = Ae^{t}(-2\sin 2t) + Ae^{t}\cos 2t + Be^{t}(2\cos 2t) + Be^{t}\sin 2t$$

$$= (B - 2A)e^{t}\sin 2t + (A + 2B)e^{t}\cos 2t$$

$$y_p'' = (B-2A)e^{t}\,2\cos 2t + (B-2A)e^{t}\sin 2t + (A+2B)e^{t}(-2\sin 2t) + (A+2B)e^{t}\cos 2t$$

$$2(B-2A)e^{t}\cos 2t + (A+2B)e^{t}\cos 2t + (B-2A)e^{t}\sin 2t - 2(A+2B)e^{t}\sin 2t$$

$$-3\left[(B-2A)e^{t}\sin 2t + (A+2B)e^{t}\cos 2t\right]$$

$$-4\left[Ae^{t}\cos 2t + Be^{t}\sin 2t\right] = -8e^{t}\cos 2t$$

$$2(B-2A) + (A+2B) - 3(A+2B) - 4A = -8 \quad \text{—(1)}$$

$$(B-2A) - 2(A+2B) - 3(B-2A) - 4B = 0 \quad \text{—(2)}$$

$$\left.\begin{aligned} 10A + 2B &= 8 \\ 2A - 10B &= 0 \end{aligned}\right\} \Rightarrow \boxed{A = \frac{10}{13},\ B = \frac{2}{13}}$$

$$\boxed{y_p = \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t}$$

Ex Find a particular solution of:

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

Sol We can take:

$y'' - 3y' - 4y = 3e^{2t}$: $y_p = -\frac{1}{2}e^{2t}$

$y'' - 3y' - 4y = 2\sin t$: $y_p = \frac{3}{17}\cos t - \frac{5}{17}\sin t$

$y'' - 3y' - 4y = -8e^{t}\cos 2t$: $y_p = \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$

Thus: The particular solution of the given O.D.E is:

$$\boxed{y_p = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t}$$

Ex Find a particular solution of $y'' + 4y = 3\cos 2t$

Sol First solve $y'' + 4y = 0$ to find y_c:

$$r^2 + 4 = 0 \Rightarrow r = \mp 2i$$

$y_1 = e^{0t}\cos 2t = \cos 2t$

$y_2 = e^{0t}\sin 2t = \sin 2t$

$$\boxed{y_c = C_1\cos 2t + C_2\sin 2t}$$

$y_p = A\cos 2t + B\sin 2t$ $\Leftarrow$ we must multiply by t

$$\boxed{y_p = At\cos 2t + Bt\sin 2t}$$

$y_p' =$

$y_p'' =$ ...... $A = 0$, $B = \frac{3}{4}$ $\Rightarrow$ $\boxed{y_p = \frac{3}{4}t\sin 2t}$

Ex $\quad y'' + 4y = t\sin t + 2\cos t$

$$y_c = c_1 \cos 2t + c_2 \sin 2t$$

$$y_p = (A_0 t + B_0)\sin t + (A_1 t + B_1)\cos t$$

....

Ex $\quad y'' + y' = t^2$

y_c: $r^2 + r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0,\ r = 1$

$$y_c = c_1 e^{0t} + c_2 e^{t}$$

$$\boxed{y_c = c_1 + c_2 e^{t}}$$

$$y_p = t(At^2 + Bt + C)$$

$$y_p = At^3 + Bt^2 + Ct$$

⋮

Ex $\quad y'' - 3y' - 4y = 3e^{4t}$

$$y_c = c_1 e^{4t} + c_2 e^{-t}$$

$$\boxed{y_p = Ate^{4t}}$$

Ex $\quad y'' - 3y' - 4y = te^{4t}$

$$y_c = c_1 e^{4t} + c_2 e^{-t}$$

$$\boxed{y_p = t(At + B)e^{4t}}$$

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Ex Find a particular solution for

$$y'' + 2y' + y = t e^{-t}$$

Sol y_c :

$$y'' + 2y' + y = 0$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)(r+1) = 0$$

$$r = -1 \implies y_1 = e^{-t}$$

$$y_2 = t e^{-t}$$

$$\boxed{y_c = c_1 e^{-t} + c_2 t e^{-t}}$$

$$y_p = (A\cancel{t} + B) e^{-t}$$

$$\boxed{\cancel{y_p = t(At + B) e^{-t}}} \Rightarrow \text{No} \implies y_p = t^2 (At + B) e^{-t}$$

$$\boxed{y_p = (At^3 + Bt^2) e^{-t}} \Rightarrow \text{Yes}$$

$$y_p' =$$

$$y_p''$$

Sec 3.7 Variation of Parameters:

$$\boxed{y'' + p(t)\,y' + q(t)\,y = g(t)}$$

First: solve $y'' + p(t)\,y' + q(t)\,y = 0$

$$y_c = c_1 y_1 + c_2 y_2$$

$$y_p = -y_1 \int \frac{y_2\, g}{W(y_1, y_2)}\, dt + y_2 \int \frac{y_1\, g}{W(y_1, y_2)}\, dt$$

The general solution is $y = y_c + y_p$

ex Solve: $y'' + 4y = 3\csc t$

sol: $y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow$
$r = \mp 2i$.

$y_1 = \cos 2t$
$y_2 = \sin 2t$ $\Rightarrow \boxed{y_c = c_1 \cos 2t + c_2 \sin 2t}$

$$W(y_1, y_2) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t - -2\sin^2 2t = 2[\cos^2 2t + \sin^2 2t] = 2(1) = 2$$

$$y_p = -\cos 2t \int \frac{\sin 2t\,(3\csc t)}{2}\, dt + \sin 2t \int \frac{\cos 2t\,(3\csc t)}{2}\, dt$$

$$= -\cos 2t\; 3\int \frac{2\sin t\cos t}{2\sin t}\, dt + \frac{3\sin 2t}{2} \int \frac{1 - 2\sin^2 t}{\sin t}\, dt$$

$$= -3\cos 2t \int \cos t \, dt + \frac{3}{2}\sin 2t \int (\csc t - 2\sin t)\, dt$$

$$= -3\cos 2t \sin t + \frac{3}{2}\sin 2t \left[\ln|\csc t - \cot t| + 2\cos t \right]$$

$$\boxed{y_p = -3\cos 2t \sin t + \frac{3}{2}\sin 2t \ln|\csc t - \cot t| + 3\sin 2t \cos t}$$

Thus the general solution is:

$$y = y_c + y_p$$

$$y = c_1 \cos 2t + c_2 \sin 2t + -3\cos 2t \text{ ------}$$

Notes ① If $ay'' + by' + cy = g(t)$

first $\Downarrow$ $y'' + \frac{b}{a} y' + \frac{c}{a} y = \frac{g(t)}{a}$ قبل تطبيق V.P.

② If $g(t) = x^{-2}$ we can not use U.C.

If $g(t) = \tan x$ " " " " U.C.

If $g(t) = \sec x, \cot x, \frac{e^t}{t} = t^{-1}e^t$ ...

Use V.P.

If $y(t_0) = y_0$
$y'(t_0) = y_0'$

يتم التعويض في آخر حل وهو
$y = y_c + y_p$

Ex $y'' + y = \cos t \qquad y(0) = 0 \ , \ y'(0) = 0$

sol $y_c : \ r^2 + 1 = 0 \Rightarrow r = \mp i \Rightarrow y_1 = \cos t$

$y_2 = \sin t$

$$\boxed{y_c = c_1 \cos t + c_2 \sin t}$$

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$y_p = -\cos t \int \frac{\sin t \cos t}{1} \, dt + \sin t \int \frac{\cos t \cos t}{1} \, dt$$

$$= -\cos t \int \frac{\sin 2t}{2} \, dt + \sin t \int \cos^2 t \, dt$$

$$= -\cos t \ \frac{-\cos 2t}{4} + \sin t \int \frac{1 + \cos 2t}{2} \, dt$$

$$= + \frac{\cos t \cos 2t}{4} + \sin t \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]$$

$$= \frac{\cos t \cos 2t + \sin t \sin 2t}{4} + \frac{t}{2} \sin t$$

$\cos(A - B) =$

$\cos A \cos B + \sin A \sin B$

$$= \frac{\cos(2t - t)}{4} + \frac{t}{2} \sin t$$

$$\boxed{y_p = \frac{\cos t}{4} + \frac{t}{2} \sin t}$$

!!!

$$y_G = c_1 \cos t + c_2 \sin t + \frac{\cos t}{4} + \frac{t}{2} \sin t$$

$y(0) = \cdot ---$ $\rightarrow$ ①

$y'(0) = \cdot ---$ $\rightarrow$ ②

$\Rightarrow c_1, c_2$