Differential Equations

Chapter One : Introduction

Def. A differential Equation is an equation that relates some function with its derivatives.

Ordinary differential equation (O.D.E)

If the unknown function depends on a single independent variable and the derivatives are ordinary

ex $y(t)$:

$$y' = 7y^2 t^2$$

Partial differential equation (P.D

If the unknown function depends on several independent variables and the derivatives are partial.

ex $u(x,t)$:

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial t^2} = 2xt$$

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* Classification of Differential equations (O.D.E)

1. The Order of O.D.E : It is the highest derivative in the O.D.E

ex
$$3y' + 3y - 4y''' = 3t \Rightarrow \text{order} = 3$$
$$2y'''' + x(y')^5 = 3x^2 \Rightarrow \text{order} = 4$$
$$2y' + 3xy = 0 \Rightarrow \text{order} = 1$$
$$3(y''')^7 - 2y = 3t \Rightarrow \text{order} = 3$$

Linear and non Linear O.D.E

The O.D.E is Linear if it is a Linear function of the variables $y, y', \dots, y^{(n)}$. The general form of the Linear O.D.E of order n is: $f_1(t)\,y^{(n)} + f_2(t)\,y^{(n-1)} + \dots + f_n(t)\,y = g(t)$.

ex
$y'' + 2t^2 y = 0$ order = 2 / Linear

$y''\sin t - t y y' = 2t$ order = 2 / non Linear.

$(y')^2 - 2y = 3t$ order = 1 / not linear

$y''' + \sin t\, y'' = t$ order = 3 / Linear.

3) **Initial value problem:** It is an O.D.E with initial conditions

ex
$$\underbrace{y' + \frac{2}{t}y = 4t}_{O.D.E} \quad \text{and} \quad \underbrace{y(t) = 2}_{\text{Initial Condition}}$$

$\rightarrow$ I.V.P $\leftarrow$

4) **A solution of O.D.E:** Is the function that satisfies the equation.

ex Is $y = \cos t$ a solution of $y + y'' = 0$.

sol $y = \cos t \Rightarrow y' = -\sin t \Rightarrow y'' = -\cos t$

$y + y'' \overset{?}{=} 0$

$\cos t + -\cos t = 0$ Yes

ex Is $y = 2 + 3e^{2t}$ a solution of $y' = 2y - 4$.

sol $y' = 6e^{2t}$

$y' \overset{?}{=} 2y - 4$

$6e^{2t} \overset{?}{=} 2(2+3e^{2t}) - 4$

$6e^{2t} = 4 + 6e^{2t} - 4$ Yes

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Chapter 2 : First Order Differential Equations (y')

1 Linear first Order O.D.E :

The general form of this type is :

$$\boxed{y' + p(t)\, y = g(t)}$$

The solution of this equation is by finding the Integrating factor

The method

1. Write the equation in the general form

$$y' + p(t)\, y = g(t)$$

2. find the Integrating factor $\mu(t)$

$$\mu(t) = e^{\int p(t)\,dt}$$

3. $$\boxed{y\, \mu(t) = \int \mu(t)\, g(t)\, dt}$$

4. find y

5. find C if the problem is IVP.

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ex Solve the O.D.E : $y' + 2y = 3$

sol $y' + 2y = 3$

$p(t) = 2 \qquad g(t) = 3$

$$\mu(t) = e^{\int 2\,dt} = e^{2t}$$

$$y\,\mu(t) = \int \mu(t)\, g(t)\, dt$$

$$y\, e^{2t} = \int 3 e^{2t}\, dt$$

$$y\, e^{2t} = \frac{3}{2} e^{2t} + C$$

$$\boxed{y = \frac{3}{2} + \frac{C}{e^{2t}}}$$

ex Solve the I.V.P (3)

$$t^3 y' + 2t^2 y = t^2 \sin t \quad \text{and} \quad y(\pi)=0$$

sol $y' + \frac{2}{t} y = \frac{\sin t}{t}$

$\boxed{p(t) = \frac{2}{t}}$ $\boxed{g(t) = \frac{\sin t}{t}}$

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2$$

$$y\mu(t) = \int \mu(t) g(t)\, dt$$

$$y t^2 = \int t^2 \frac{\sin t}{t}\, dt$$

$$y t^2 = \int t \sin t\, dt$$

	D.	I.
+	t	$\sin t$
−	1	$-\cos t$
+	0	$-\sin t$

$$y t^2 = -t\cos t + \sin t + C$$

$$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}$$

now to find C use $y(\pi)=0$

$$y(\pi) = -\frac{\cos\pi}{\pi} + \frac{\sin\pi}{\pi^2} + \frac{C}{\pi^2}$$

$$0 = \frac{-(-1)}{\pi} + 0 + \frac{C}{\pi^2}$$

$\Rightarrow \boxed{C = -\pi}$

$$\boxed{\therefore\ y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} - \frac{\pi}{t^2}}$$

H.W solve $y' + \frac{1}{2} y = 2 + t$

sol $\boxed{y = 2t + C e^{-\frac{1}{2}t}}$

	D.	I.
+	$2+t$	$e^{\frac{1}{2}t}$
−	t	$2e^{\frac{1}{2}t}$
+	0	$4e^{\frac{1}{2}t}$

2 Bernolli Differential Equation

Separable

The general form of this type is :

$$\boxed{y' + p(t)y = q(t)\,y^n} \quad , n \geq 2$$

This equation is nonLinear First-order O.D.E, the solution of it is by Converting it to a Linear form.

The method

1. Write the equation in the general form

$$y' + p(t)y = q(t)\,y^n$$

2. Let $W = y^{1-n} \Rightarrow \boxed{w' = (1-n)\,y^{-n}\,y'}$

and then we have the Linear equation

$$\frac{1}{1-n} w' + p(t)w = q(t)$$

3. Solve this equation which is Linear and First order by Find the Integrating factor.

4. Let $w = y^{1-n}$ again.

$$y'\,y^{-n} + p(t)\,y^{1-n} = q(t)$$

$$\frac{w'}{1-n} + p(t)W = q(t)$$

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ex Solve $t^2 y' + 2ty - y^3 = 0$

sol $\boxed{y' + \frac{2}{t}y = \frac{1}{t^2}y^3}$

$W = y^{-2}$ $\quad W = \frac{1}{y^2}$

$$\frac{1}{-2}w' + \frac{2W}{t} = \frac{1}{t^2}$$

$$\Rightarrow \boxed{w' - \frac{4}{t}w = \frac{-2}{t^2}}$$

$P(t) \qquad g(t)$

$$\mu(t) = e^{\int -\frac{4}{t}} = t^{-4}$$

$$w\mu(t) = \int \mu(t)\,g(t)\,dt$$

$$wt^{-4} = \int t^{-4}\frac{(-2)}{t^2}\,dt$$

$$wt^{-4} = \frac{-2t^{-5}}{-5} + C$$

$$wt^{-4} = \frac{2}{5}t^{-5} + C$$

$$w = \frac{2}{5}t^{-1} + ct^4 .$$

$$\frac{1}{y^2} = \frac{2}{5t} + ct^4 \Rightarrow$$

$$\frac{1}{y^2} = \frac{2+5ct^5}{\ldots} \Rightarrow y = \mp\sqrt{\frac{5t}{\ldots}}$$

Ex (H.W) Solve $y' + \frac{4}{t}y = t^3 y^2$, $t > 0$, $y(2) = -1$.

Sol

$$y' + \frac{4}{t}y = t^3 y^2$$

$$w = y^{-1} \qquad w = y^{1-n}$$

$$\frac{1}{-1}w' + \frac{4}{t}w = t^3$$

$$\Rightarrow \boxed{w' - \frac{4}{t}w = -t^3}$$ now solve it by I.F.

$$\underbrace{\qquad}_{p(t)} \qquad \underbrace{\qquad}_{g(t)}$$

$$\mu(t) = e^{\int -\frac{4}{t}dt} = e^{-4\ln t} = t^{-4}$$

$$w t^{-4} = \int -t^{-4} t^3 \, dt$$

$$w t^{-4} = -\int t^{-1} \, dt$$

$$w t^{-4} = -\ln|t| + C$$ but $t > 0$

$$w t^{-4} = -\ln t + C$$

$$\Rightarrow w = Ct^4 - t^4 \ln t$$

$$\frac{1}{y} = Ct^4 - t^4 \ln t$$

$$y = \frac{1}{Ct^4 - t^4 \ln t}$$

to find C

$$y(2) = \frac{1}{C(2)^4 - (2)^4 \ln 2}$$

$$-1 = \frac{1}{16C - 16\ln 2}$$

$$16\ln 2 - 16C = 1 \Rightarrow \boxed{\cancel{C = \ln 2}}$$

$$\frac{16\ln 2 - 1}{16} = C = \ln 2 - \frac{1}{16}$$

$$y = \frac{1}{\left(\ln 2 - \frac{1}{16}\right)t^4 - t^4 \ln t}$$

3) Separable Equations

If the O.D.E is first order then it is a function of the form $y' = f(x,y)$

$\Rightarrow \boxed{\frac{dy}{dx} = f(x,y)} \equiv \boxed{M(x,y) + N(x,y)\frac{dy}{dx} = 0}$

Linear: solve use I.F.

non Linear

Bernolli : Convert to Linear.

Separable : If $M(x,y)$ is Just $M(x)$ and $N(x,y)$ is Just $N(y)$

$\Rightarrow \boxed{M(x) + N(y)\frac{dy}{dx} = 0}$ Is a separable eq.

exact

$M(x)\,dx + N(y)\,dy = 0$

If the O.D.E is of the form $M(x) + N(y)\frac{dy}{dx} = 0 \Longrightarrow$ It is separable

$y' = \frac{\boxed{M(x)}}{\boxed{N(y)}}$ or, $y' = \boxed{M(x)}\,\boxed{M(y)}$

ex solve $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

sol $(1-y^2)\,dy = x^2\,dx$

$$\boxed{y - \frac{y^3}{3} = \frac{x^3}{3} + C}$$

ex $e^{-3x} y y' = 1$, $y(0) = -1$

sol $e^{-3x} y \frac{dy}{dx} = 1 \Rightarrow y\,dy = e^{3x}\,dx$

$\Rightarrow \frac{y^2}{2} = \frac{e^{3x}}{3} + C$

$\frac{1}{2} = \frac{1}{3} + C \Rightarrow \boxed{C = \frac{1}{6}}$

$y^2 = \frac{2}{3}e^{3x} + \frac{1}{3}$

$y = \mp\sqrt{\frac{2}{3}e^{3x} + \frac{1}{3}}$

H.w

$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$

$\frac{dy}{dx} = \frac{2-e^x}{3+2y}$ $y(0) =$

$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$

$\frac{dy}{dx} = \frac{2\cos 2x}{3+2y}$ $y($

4 Exact Equation:

Revision: If f is a function of two variables x, y

i.e $f(x,y)$

f_x is $\frac{\partial f}{\partial x}$ in which y is a constant

f_y is $\frac{\partial f}{\partial y}$ in which x is a constant

ex $f(x,y) = x\cos y + x^2 e^y$

$f_x = \cos y + 2x e^y$

$f_y = -x\sin y + x^2 e^y$

ex $f(x,y) = 2xy + e^{3xy}$

$f_x = 2y + 3y e^{3xy}$

$f_y = 2x + 3x e^{3xy}$

Thm The differential equation $M(x,y) + N(x,y)y' = 0$ in which M, N, M_y, N_x are all continuous in the region $\alpha < x < \beta$, $\gamma < y < \delta$

is called exact if and only if

$M_y = N_x$

That is there exists a function $\psi(x,y)$ in which

$\psi_x = M(x,y)$

$\psi_y = N(x,y)$

The method for solving the exact O.D.E

$M(x,y) + N(x,y)\, y' = 0$

$\boxed{1}$ Make sure that the equation is exact i.e $M_y = N_x$

$\boxed{2}$ find ψ in which :

$$\boxed{\begin{matrix}\psi_x \\ M\end{matrix}} + \boxed{\begin{matrix}\psi_y \\ N\end{matrix}} \begin{matrix} y' = 0 \\ y' = 0\end{matrix}$$

$\boxed{3}$ $\psi_x = M \implies$

$\psi(x,y) = \int M\, dx + h(y)$

$\boxed{4}$ $\psi_y = \dots\dots + h'(y) = N$

Then find $h(y)$

$\boxed{5}$ take the function $\psi(x,y) = 0$

ex Solve the O.D.E

$\boxed{2x+y^2} + \boxed{2xy}\, y' = 0$

sol $M(x,y) + N(x,y)\, y' = 0$

$M = 2x+y^2 \implies M_y = 2y$
$N = 2xy \implies N_x = 2y$ $\implies$ exact.

$\exists \psi(x,y)$ such that

$\psi_x = M \implies \psi_x = 2x+y^2$
$\psi_y = N \implies \psi_y = 2xy$

Now $\psi_x = 2x+y^2$

$\psi = x^2 + y^2 x + h(y)$

$\psi_y = 2xy + h'(y) = 2xy \implies$

$h'(y) = 0 \Rightarrow h(y) = C$

$\psi(x,y) = x^2 + y^2 x + C$

$0 = x^2 + y^2 x + C$

$\Rightarrow \boxed{x^2 + y^2 x = C}$

ex $\underbrace{y\cos x + 2xe^y} + \underbrace{(\sin x + x^2 e^y + 1)}\, y' = 0$

sol $M(x,y) + N(x,y)\, y' = 0$

$$\boxed{\begin{aligned} M_y &= \cos x + 2xe^y \\ N_x &= \cos x + 2xe^{y-1} \Rightarrow \text{exact} \end{aligned}}$$

$M = \psi_x = y\cos x + 2xe^y \quad , \quad N = \psi_y = \sin x + x^2 e^y - 1$

$\psi(x,y) = \int (y\cos x + 2xe^y)\, dx + h(y)$

$$\boxed{\psi(x,y) = y\sin x + x^2 e^y + h(y)}$$

$\psi_y = \sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1$

$\Rightarrow h'(y) = -1$

$\Rightarrow h(y) = -y + C$

$\therefore \psi(x,y) = y\sin x + x^2 e^y - y + C$ Let $\psi(x,y) = 0$

$\Rightarrow \boxed{y\sin x + x^2 e^y - y = C}$

H.w solve $y\cos x + 2xe^y + (\sin x + x^2 e^y)y' = 0$

sol $\boxed{y\sin x + x^2 e^y = C}$

any separable eq. is exact !??!

H.w $x^2 y^3 + x^3(\cancel{1+}y^2)\, y' = 0$

$3x^2 y + y^2 x + (x^3 + xy^2)y' = 0$

Ex② $\left(y\cos x + 2xe^{y}\right)dx + \left(\sin x + x^2 e^{y} - 1\right)dy = 0$

sol $\underbrace{\left(y\cos x + 2xe^{y}\right)}_{M} + \underbrace{\left(\sin x + x^2e^{y} - 1\right)}_{N}\frac{dy}{dx} = 0$

$$\left.\begin{aligned} M_y &= \cos x + 2xe^{y} \\ N_x &= \cos x + 2xe^{y} \end{aligned}\right\} M_y = N_x \implies \text{exact.}$$

$$\Psi = \int\left(y\cos x + 2xe^{y}\right)dx = \boxed{y\sin x + x^2e^{y}} + h(y)$$

$$\Psi = \int\left(\sin x + x^2e^{y} - 1\right)dy = \boxed{y\sin x + x^2e^{y}} - y + h(x)$$

$$\Downarrow$$

$$h(y) = -y$$
$$h(x) = 0$$

Thus : The solution is $\boxed{y\sin x + x^2e^{y} - y + C = 0}$

$$\equiv \boxed{y\sin x + x^2e^{y} - y = \overset{*}{C}}$$

or $\left(y\cos x + 2x e^{y}\right) + \left(\sin x + x^2 e^{y} - 1\right) y' = 0$

$M_y =$

$N_x =$ ———

$\psi = \int \left(y\cos x + 2x e^{y}\right) dx = y\sin x + x^2 e^{y} + h(y)$

$\psi_y = \sin x + x^2 e^{y} + h'(y)$

$= \sin x + x^2 e^{y} - 1$

Thus: $h'(y) = -1$

$h(y) = -y + c$

The solution is $\sin x + x^2 e^{y} - y + c = 0$

$\equiv \boxed{\sin x + x^2 e^{y} - y = c^*}$

Exo ① Solve: $\underbrace{y\cos x + 2xe^{y}}_{M} + \underbrace{\sin x + x^{2}e^{y}}_{N}\, y' = 0$

Solution

$$M = y\cos x + 2xe^{y} \Longrightarrow M_y = \cos x + 2xe^{y}$$
$$N = \sin x + x^{2}e^{y} \Longrightarrow N_x = \cos x + 2xe^{y}$$
$$\Longrightarrow M_y = N_x \Longrightarrow \text{exact.}$$

①
نكامل M بالنسبة إلى x ونعتبر y ثابت

$$\Psi = \int \left(y\cos x + 2xe^{y}\right) dx$$
$$= \boxed{y\sin x + x^{2}e^{y} + h(y)}$$

②
نشتق Ψ بالنسبة إلى y ونعتبر x ثابتة

$$\Psi_y = \sin x + x^{2}e^{y} + h'(y) = N$$
$$= \sin x + x^{2}e^{y}$$
$$\Longrightarrow h'(y) = 0 \Longrightarrow h(y) = c$$

③
$\Psi_y = N$

Thus:

$$\boxed{y\sin x + x^{2}e^{y} + c = 0}$$
$$\equiv$$
$$\boxed{y\sin x + x^{2}e^{y} = C^{*}}$$

Solve $\underbrace{y\cos x + 2x e^{y}}_{M} + \underbrace{\sin x + x^{2} e^{y}}_{N}\, y' = 0$

$$\Psi = \int \left(y\cos x + 2x e^{y} \right) dx$$

$$= y\sin x + x^{2} e^{y} + h_1(y)$$

تكامل M بالنسبة لـ x

تعتبر الثابت هو y

$$\Psi = \int \left(\sin x + x^{2} e^{y} \right) dy$$

$$= (\sin x)y + x^{2} e^{y} + h_2(x)$$

تكامل M بالنسبة لـ y

تعتبر الثابت هو x

Thus: $h_1(y) = 0$ and $h_2(x) = 0$ So!

$$y\sin x + x^{2} e^{y} + C = 0$$

$$\equiv \boxed{y\sin x + x^{2} e^{y} = C^{*}}$$

① $(y\cos x - x\sin x)\,dx + \sin x\, dy = 0$, $y(\frac{\pi}{2}) = 2$ $\boxed{\text{Lin- exact}}$

② $x y' + 3(y + x^{2}) = \frac{\sin x}{x}$ $\boxed{\text{Linear}}$

③ $x^{2} y' = \frac{4x^{2} - x - 2}{(x+1)(y+1)}$, $y(1) = 1$ $\boxed{\text{sep.}}$

④ $\left(y e^{xy} - \frac{1}{y} \right) dx + \left(x e^{xy} + \frac{x}{y^{2}} \right) dy = 0$, $y(1) = 1$ $\boxed{\text{exact.}}$

- equation $(3xy + y^2) + (x^2 + xy)y' = 0$ exact.

$M_y = 3x + 2y$

$N_x = 2x + y$

since $M_y \neq N_x \Rightarrow$ the equation is not exact.

2.4

Differences Between Linear and Non Linear Equations:

Existence and Uniqueness of Solutions

$\boxed{1}$ for Linear equation: $y' + p(t)y = g(t)$, $y(t_0) = y_0$

Thm: If $p(t)$, $g(t)$ are continuous on $\alpha < t < \beta$ containing the point t_0, then there exists a unique function (solution) $y = \phi(t)$ that satisfies the differential equation: $y' + p(t)y = g(t)$ ~~for each t in I~~, and that also satisfies the initial condition $y(t_0) = y_0$

ex find an interval on which the I.V.P

$ty' + 2y = 4t^2$, $y(1) = 2$

Is the equation $(3xy + y^2) + (x^2 + xy)y' = 0$ exact.

<u>sol</u> $M_y = 3x + 2y$

$N_x = 2x + y$

since $M_y \neq N_x \Rightarrow$ the equation is not exact.

<u>sec. 2.4</u>

Differences Between Linear and NonLinear Equations:

Existence and Uniqueness of Solutions

1. for Linear equation: $y' + p(t)y = g(t)$, $y(t_0) = y_0$

Thm: If $p(t), g(t)$ are continuous on $\alpha < t < \beta$ containing the point t_0, then there exists a unique function (solution) ~~$y = \phi(t)$~~ that satisfies the differential equation: $y' + p(t)y = g(t)$ ~~for each t in I~~, and that also satisfies the initial condition $y(t_0) = y_0$

<u>ex</u> find an interval on which the I.V.P

$ty' + 2y = 4t^2$, $y(1) = 2$

has a unique solution

<u>sol</u> $y' + \frac{2}{t}y = 4t$

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