Fourier transform

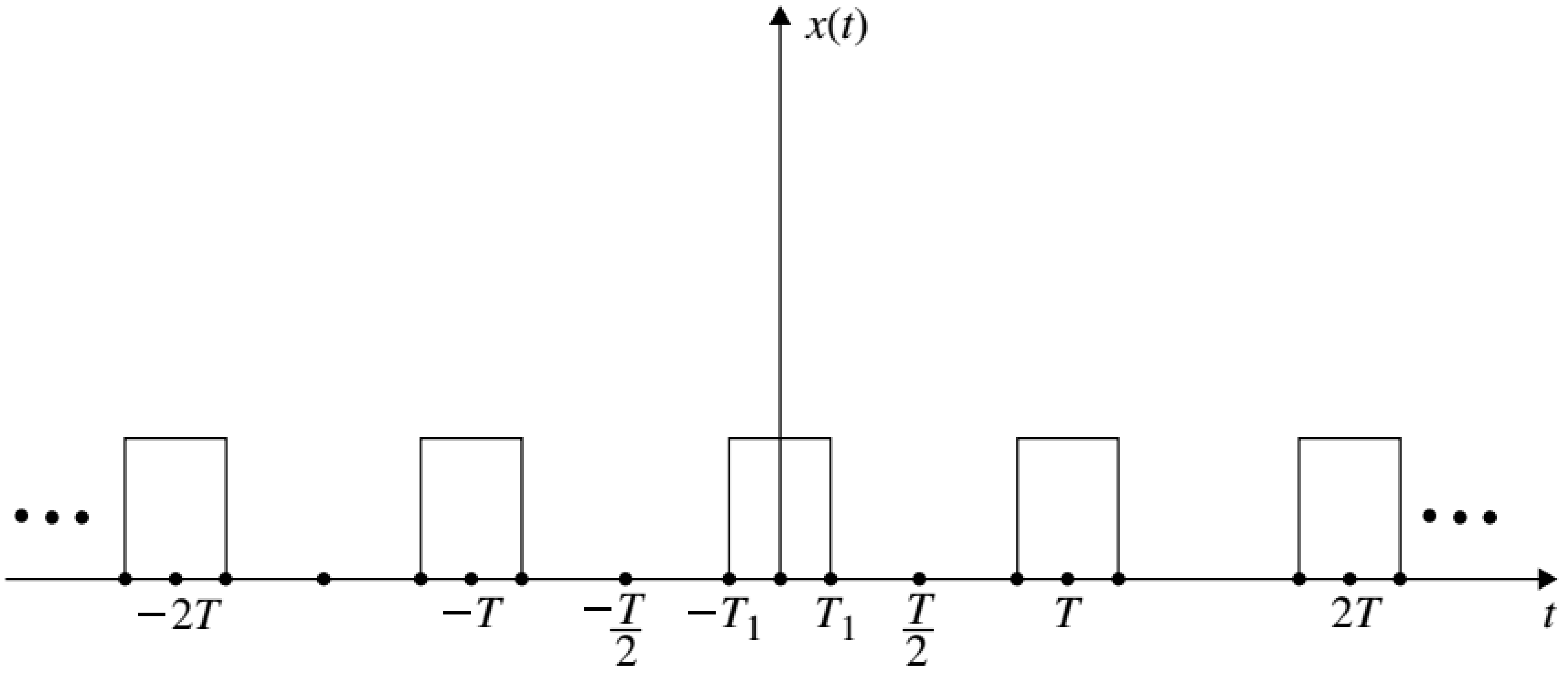
CH5

- Periodic signal can be presented by Fourier series .
- Can we present a Non periodic signal (aperiodic signal) as Fourier series ?
- Try this : take a peridic signal and make T_o-> infinity ($T_o \rightarrow \infty$)

$T \uparrow \; f \downarrow \Rightarrow$ Fourier series becomes an Intgral

Consider the periodic signal $x(t)$ defined as follows:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$$

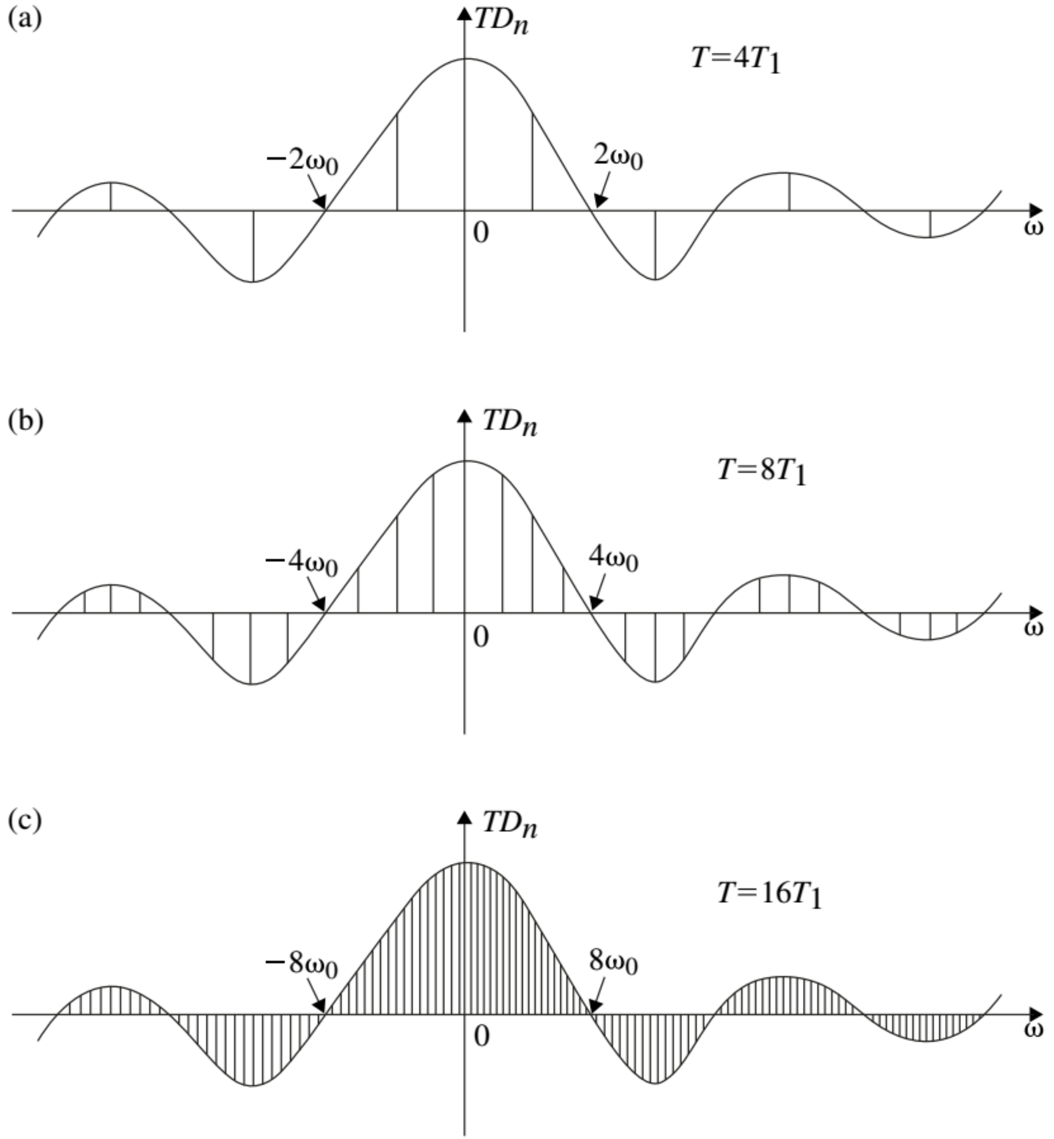


Fourier series coefficients D_n can be determined as

$$D_n = \frac{2\sin(n\omega_0 T_1)}{(n\omega_0 T)}$$

where $\omega_0 = \frac{2\pi}{T}$. The Fourier series coefficients TD_n are obtained as

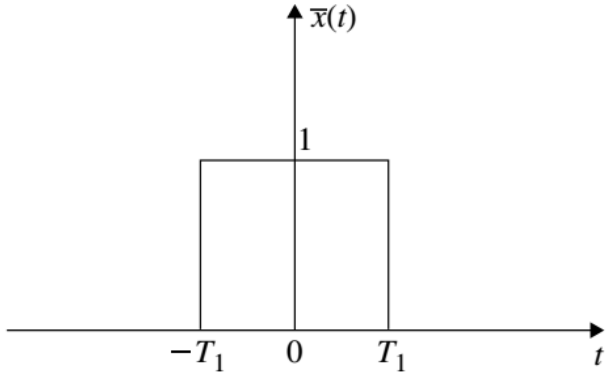
$$TD_n = \frac{2\sin(n\omega_0 T_1)}{(n\omega_0)}$$


(a)
TDn
T=4T1
−2ω0
2ω0
0
ω
(b)
TDn
T=8T1
−4ω0
4ω0
0
ω
(c)
TDn
T=16T1
−8ω0
8ω0
0
ω

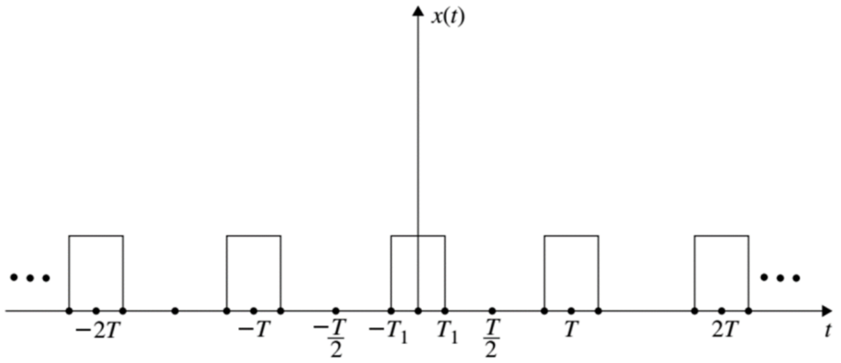
- it is evident that as T increases (the fundamental frequency $\omega 0 = 2\pi/T$ decreases),
- the samples of TDn become closer and closer.
- As T becomes very large, the original periodic square wave becomes a rectangular pulse. As $T \rightarrow \infty$, TDn becomes continuous.

Let $\bar{x}(t)$ be a non-periodic square wave as represented

$$\bar{x}(t) = 0 \qquad |t| > T_1$$



The periodic signal $x(t)$ formed by repeating $\bar{x}(t)$ with fundamental period T is shown



If $T \to \infty$

$$\underset{T \to \infty}{Lt}\, x(t) = \bar{x}(t)$$

The Fourier series coefficients of a periodic signal are written as

$$D_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t}\, dt$$

The periodic signal $x(t)$ can be expressed in Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} D_n\, e^{jn\omega_0 t}$$

$$Tx(t) = \sum_{n=-\infty}^{\infty} TD_n\, e^{jn\omega_0 t}$$

$$
\begin{aligned}
X(n\omega_0) &= TD_n \\
&= \int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}\,dt \\
x(t) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} TD_n\, e^{jn\omega_0 t} \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(n\omega_0)e^{jn\omega_0 t}\omega_0
\end{aligned}
$$

As $T \to \infty$, $\omega_0 = \frac{2\pi}{T} \to 0$ and $n\omega_0 = \omega$ which is continuous.

the summation becomes an integration.

$$
\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt} \quad \text{for all } \omega
$$

called Fourier transform pair

analysis equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \quad \text{for all } \omega$$

synthesis equation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad \text{for all } t$$

$$X(j\omega) = F[x(t)]$$

$$x(t) \overset{\text{FT}}{\longleftrightarrow} X(j\omega)$$

$$x(t) = F^{-1}[X(j\omega)]$$

$$X(j\omega) \overset{\text{IFT}}{\longleftrightarrow} x(t)$$

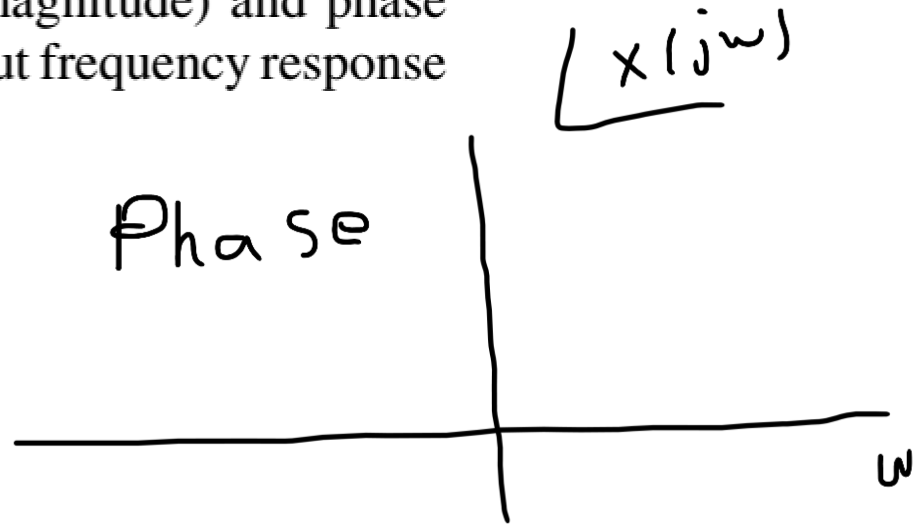
- The time function x(t) is always denoted by a lower case letter and the frequency function X(jω) by a capital letter.
- Further, when x(t) is Fourier transformed, it becomes complex and so it is denoted as X(jω).
- In some literature, X(jω) is also represented simply as X(ω).

Fourier Spectra

The Fourier transform of $X(j\omega)$ of $x(t)$ is, in general, complex and can be expressed as

$$X(j\omega) = |X(j\omega)| \; \angle X(j\omega)$$

The plot of $|X(j\omega)|$ versus ω is called magnitude spectrum of $X(j\omega)$. The plot of $\angle X(j\omega)$ versus ω is called phase spectrum. The amplitude (magnitude) and phase spectra are together called Fourier spectrum which is nothing but frequency response of $X(j\omega)$ for the frequency range $-\infty < \omega < \infty$.

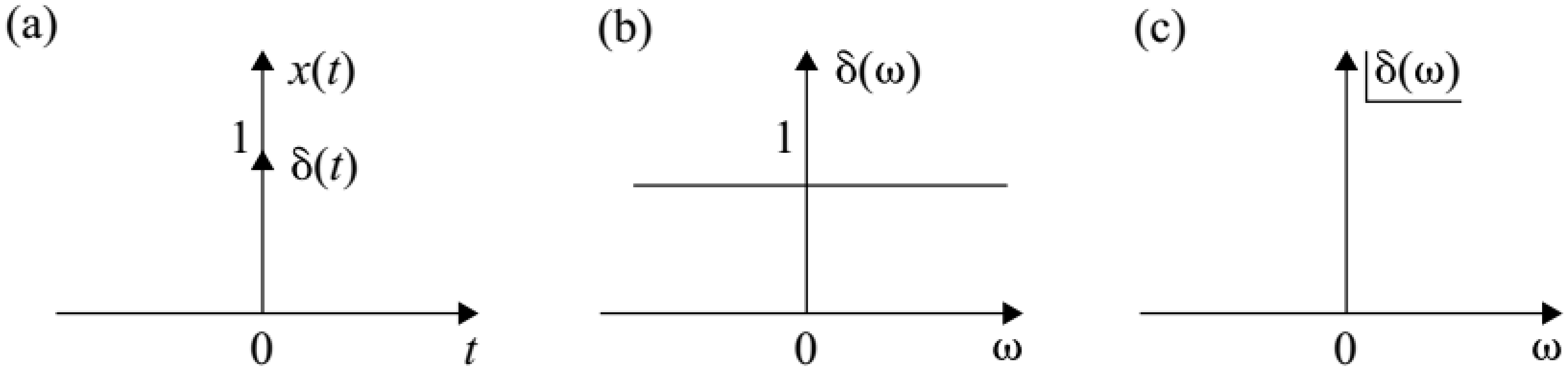


Find the Fourier transform of the following time functions and sketch their Fourier spectra (amplitude and phase).

$\boldsymbol{x(t) = \delta(t)}$

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \delta(t)\, dt \qquad\qquad [\delta(t) = 0 \quad \text{for } t \neq 0 \\
&= 1 \qquad\qquad\qquad\qquad\quad = 1 \quad \text{for } t = 0]
\end{aligned}
$$

$$
\boxed{\delta(t) \overset{\text{FT}}{\longleftrightarrow} 1}
$$



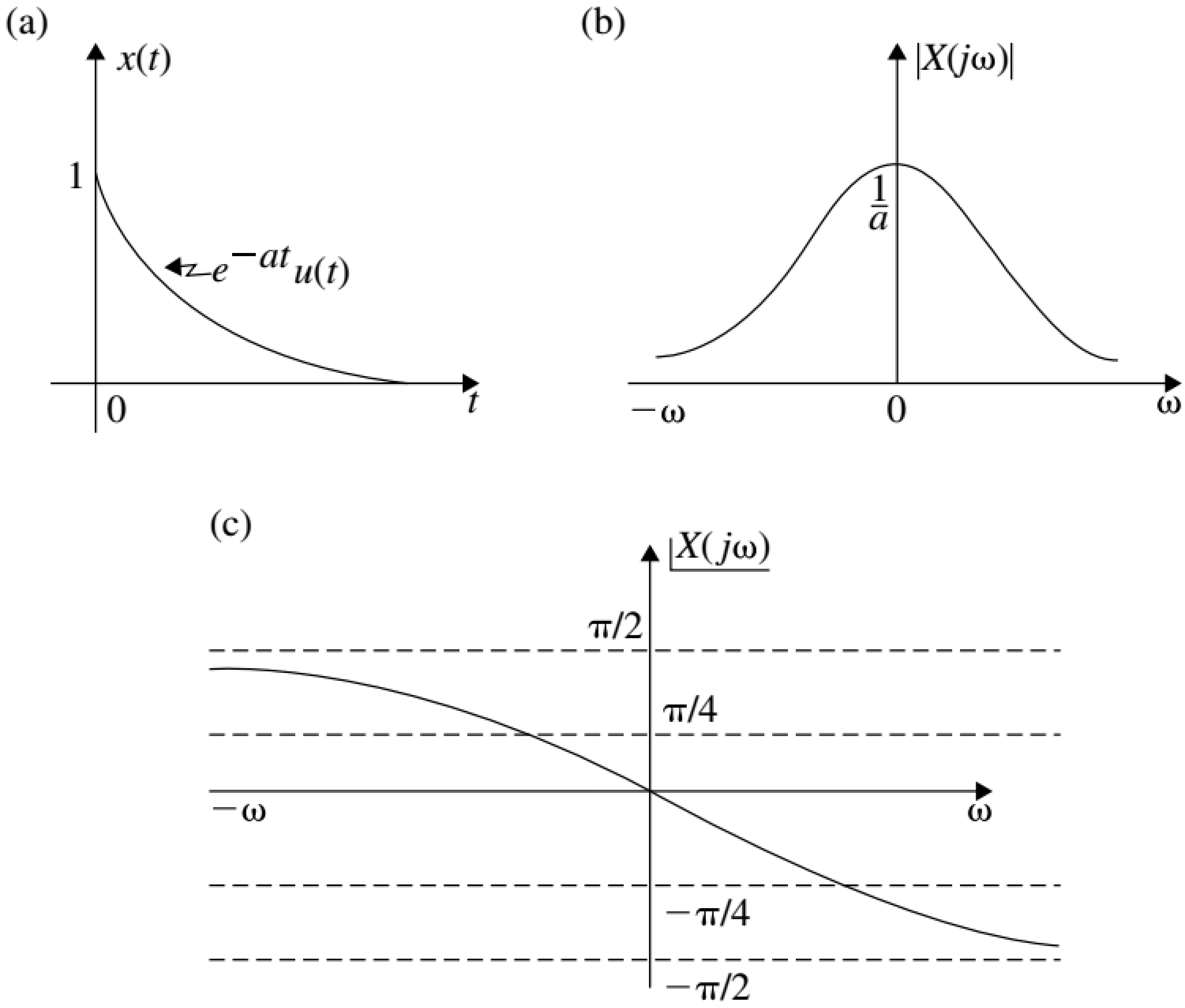
(a)
x(t)
1
δ(t)
0
t
(b)
δ(ω)
1
0
ω
(c)
δ(ω)
0
ω

$\boldsymbol{x(t) = e^{-at}u(t);\ a > 0}$

$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t}\, dt$$
$$= \int_0^\infty e^{-(a+j\omega)t}\, dt$$
$$= -\frac{1}{(a+j\omega)} \left[e^{-(a+j\omega)t} \right]_0^\infty$$

$$\boxed{X(j\omega) = \frac{1}{(a+j\omega)}}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$
$$\angle X(j\omega) = -\tan^{-1}\frac{\omega}{a}$$


(a)
x(t)
1
$e^{-at}u(t)$
0
t
(b)
|X(jω)|
$\frac{1}{a}$
−ω
0
ω
(c)
∠X(jω)
π/2
π/4
−ω
ω
−π/4
−π/2

$x(t) = e^{at}u(t);\ a > 0$

$$X(j\omega) = \int_0^\infty e^{at}e^{-j\omega t}\,dt$$

$$= \int_0^\infty e^{(a-j\omega)t}\,dt$$

$$= \frac{1}{(a-j\omega)}\left[e^{(a-j\omega)t}\right]_0^\infty$$

If the upper limit is applied to the above integral, the Fourier integral does not converge. **Hence, FT does not exist for $x(t) = e^{at}u(t)$.**

$$x(t) = e^{at}u(-t) \qquad a > 0$$

$$x(-t) = e^{-at}u(t)$$

$$\begin{aligned} F[e^{at}u(-t)] &= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt \\ &= \frac{1}{(a-j\omega)} \left[e^{(a-j\omega)t} \right]_{-\infty}^{0} \end{aligned}$$

$$\boxed{F[e^{at}u(-t)] = \frac{1}{(a-j\omega)}}$$

$$\boldsymbol{x(t) = e^{-a|t|};\ a > 0}$$

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt
\end{aligned}$$

$$\begin{aligned}
X(j\omega) &= \frac{1}{(a-j\omega)}\left[e^{(a-j\omega)t}\right]_{-\infty}^{0} - \frac{1}{(a+j\omega)}\left[e^{-(a+j\omega)t}\right]_{0}^{\infty} \\
&= \frac{1}{(a-j\omega)} + \frac{1}{(a+j\omega)}
\end{aligned}$$

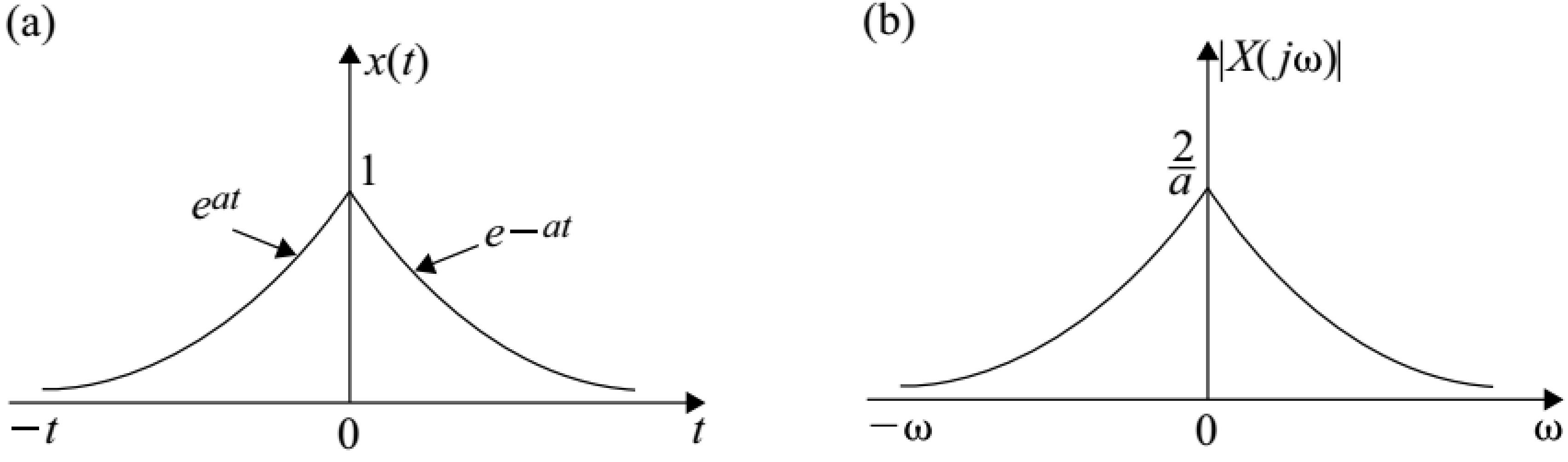
$$\boxed{X(j\omega) = \frac{2a}{a^2 + \omega^2}}$$

$$\left[e^{-a|t|}\right] \overset{\text{FT}}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

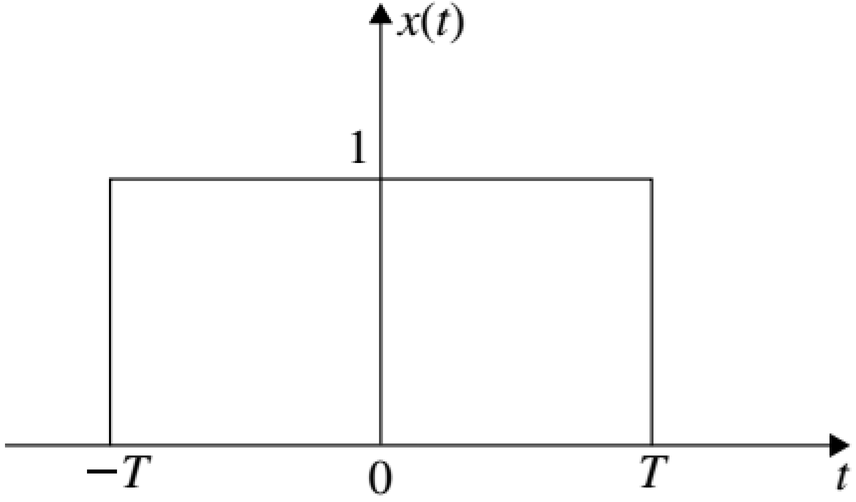
Fourier Spectra

$$|X(j\omega)| = \frac{2a}{a^2 + \omega^2}$$

$$\underline{|X(j\omega)} = 0$$



Find FT of the signal:



$$x(t) = 1 \qquad |t| \le T$$

$$\begin{aligned} X(j\omega) &= \int_{-T}^{T} 1e^{-j\omega t}\,dt \\ &= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-T}^{T} \\ &= \frac{\left[e^{j\omega T} - e^{-j\omega T}\right]}{j\omega} \\ &= \frac{2T \sin \omega T}{\omega T} = 2T\text{sinc } \omega T \end{aligned}$$

$$\boxed{X(j\omega) = 2T\text{sinc } \omega T}$$

Frequency spectra

At $\omega = 0$,

$$|X(j\omega)| = \frac{2\sin\omega T}{\omega T} = \frac{2\sin 0}{0} = 2$$

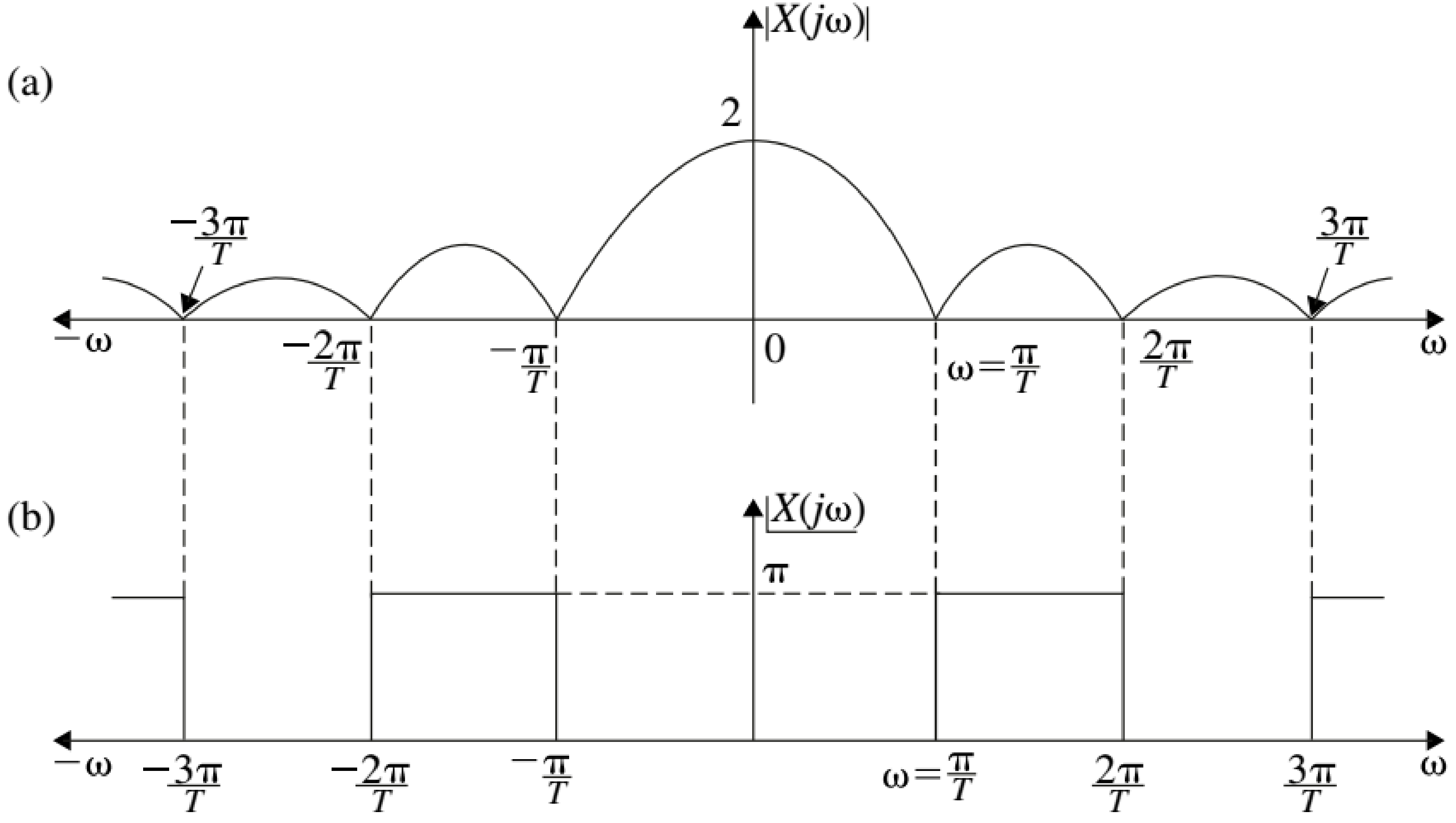
At $\omega = \pm\frac{n\pi}{T}$,

$$|X(j\omega)| = 0, \qquad \text{where } n = 1, 2, 3, \ldots$$

Phase Spectrum

$$\text{For } 0 < \omega < \frac{\pi}{2}, \qquad \underline{|X(j\omega)} = 0$$

$$\text{For } \frac{\pi}{T} < \omega < \frac{2\pi}{T}, \qquad \underline{|X(j\omega)} = \pi$$



(a)
|X(jω)|
2
−3π/T
3π/T
−ω
−2π/T
−π/T
0
ω=π/T
2π/T
ω
(b)
∠X(jω)
π
−ω
−3π/T
−2π/T
−π/T
ω=π/T
2π/T
3π/T
ω

) $x(t) = \mathrm{sgn}(t)$

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$F[\mathrm{sgn}(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= -\int_{-\infty}^{0} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-j\omega t}\, dt$$

NoT integrable

solved by the use of a **TRICK**. $x(t)$ is multiplied by $e^{-a|t|}$ $a \to 0$

$$F[e^{-a|t|}\mathrm{sgn}(t)] = \int_{-\infty}^{0} -e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$$

$$F[e^{-a|t|}\mathrm{sgn}(t)] = \int_{-\infty}^{0} -e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$F[e^{-a|t|}\mathrm{sgn}(t)] = \underset{a\to 0}{Lt}\left[\frac{-1}{a-j\omega}\left\{e^{(a-j\omega)t}\right\}_{-\infty}^{0} - \frac{1}{(a+j\omega)}\left\{e^{-(a+j\omega)t}\right\}_{0}^{\infty}\right]$$

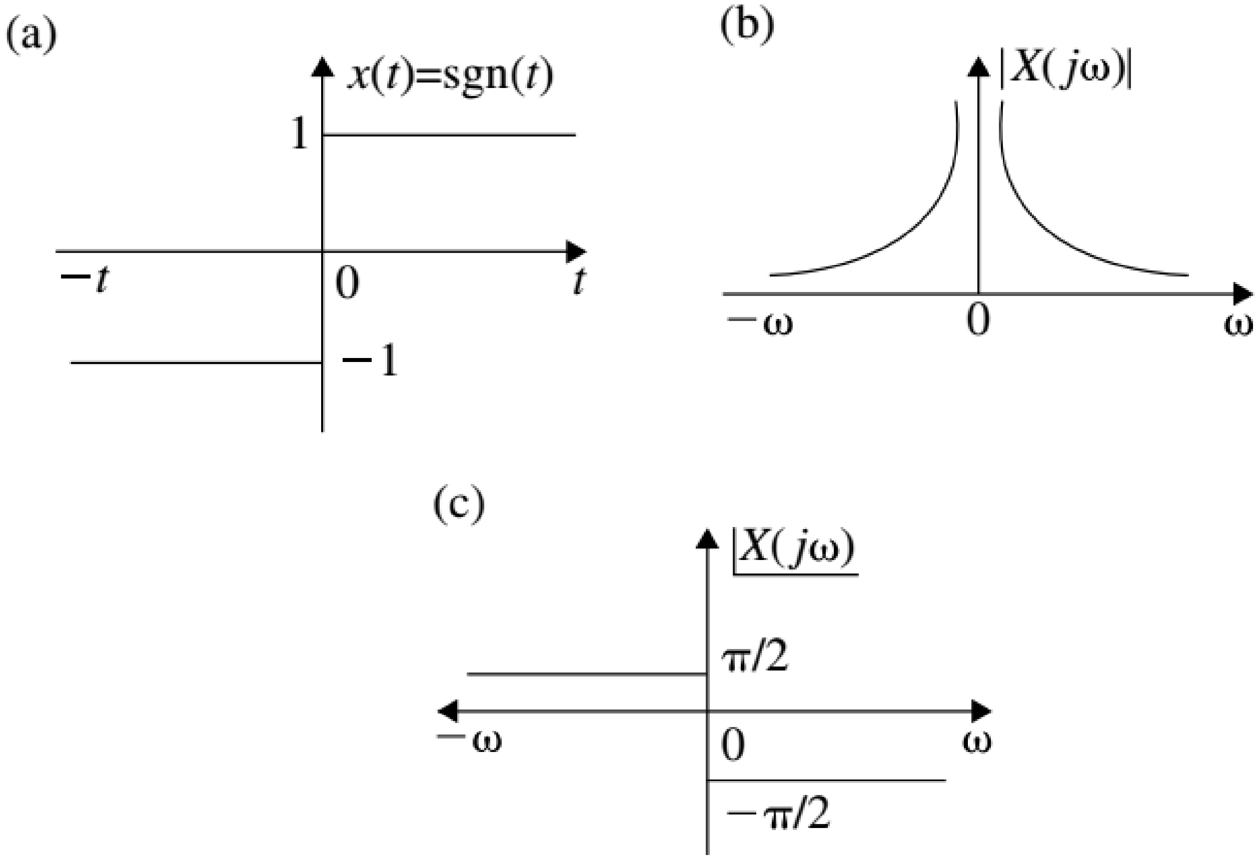
$$= \underset{a\to 0}{Lt}\left[\frac{-1}{(a-j\omega)} + \frac{1}{a+j\omega}\right] = \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}$$

$$\boxed{\mathrm{sgn}(t) \overset{\mathrm{FT}}{\longleftrightarrow} \frac{2}{j\omega}}$$

Fourier Spectra of $\mathbf{sgn}(t)$

$$X(j\omega) = \frac{2}{j\omega} = \begin{cases} \frac{2}{\omega}\angle{-90^\circ} & \omega \geq 0 \\ \frac{2}{\omega}\angle 90^\circ & \omega < 0 \end{cases}$$

$x(t) = \text{sgn}(t)$, $|X(j\omega)| = \frac{2}{\omega}$ and $\underline{|X(j\omega)}$ are represented in Fig. 6.5a, b and c, respectively.



(a)
x(t)=sgn(t)
1
−t
0
t
−1
(b)
|X(jω)|
−ω
0
ω
(c)
X(jω)
π/2
−ω
0
ω
−π/2

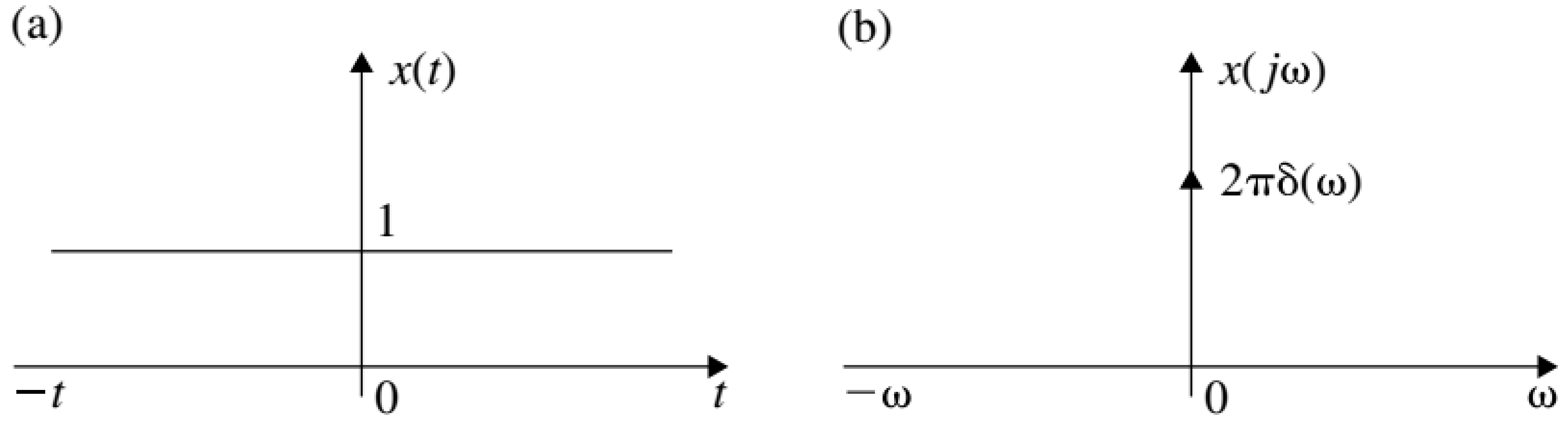
$x(t) = 1$; for all t

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}\,d\omega$$

Since $\delta(\omega)e^{j\omega t} = \delta(\omega)$,

$$\begin{aligned} F^{-1}[\delta(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\,d\omega \\ &= \frac{1}{2\pi} \qquad \text{since } \delta(\omega) = \begin{cases} 1 & \omega = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} \frac{1}{2\pi} &\overset{\text{FT}}{\longleftrightarrow} = \delta(\omega) \\ 1 &\overset{\text{FT}}{\longleftrightarrow} = 2\pi\,\delta(\omega) \end{aligned}$$


(a)
x(t)
1
−t
0
t
(b)
x(jω)
2πδ(ω)
−ω
0
ω

$x(t) = u(t)$ and $x(t) = u(-t)$

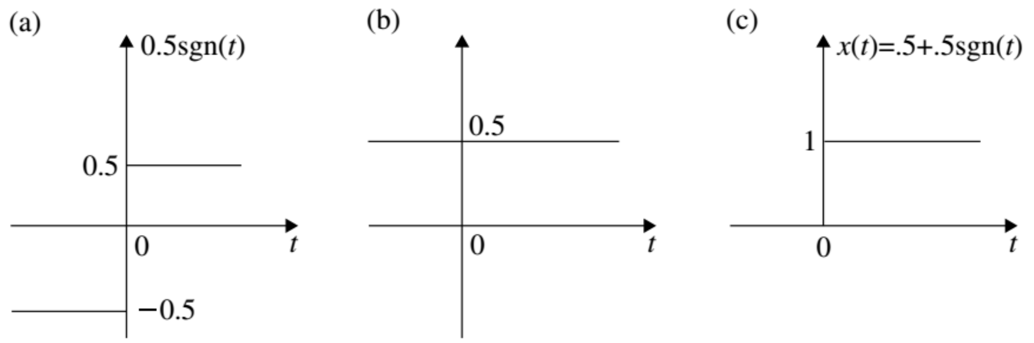
$$x(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

$$F[u(t)] = F\left[\frac{1}{2}\right] + \frac{1}{2}F\,\text{sgn}(t)$$

$$F\left[\frac{1}{2}\right] = \pi\,\delta(\omega)$$

$$F\left[\frac{1}{2}\text{sgn}(t)\right] = \frac{1}{j\omega}$$



Properties of Fourier Transform

Linearity

If

$$x_1(t) \xleftrightarrow{\text{FT}} X_1(j\omega)$$

$$x_2(t) \xleftrightarrow{\text{FT}} X_2(j\omega)$$

then

$$[A\,x_1(t) + B\,x_2(t)] \xleftrightarrow{\text{FT}} [A\,X_1(j\omega) + B\,X_2(j\omega)]$$

Time Shifting

If

$$x(t) \overset{\text{FT}}{\longleftrightarrow} X(j\omega)$$

then

$$x(t - t_0) \overset{\text{FT}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Conjugation and Conjugation Symmetry

If

$$x(t) \xleftrightarrow{\text{FT}} X(j\omega)$$

then

$$x^*(t) \xleftrightarrow{\text{FT}} X^*(-j\omega)$$

Differentiation in Time

If

$$x(t) \stackrel{\text{FT}}{\longleftrightarrow} X(j\omega)$$

then

$$\frac{dx(t)}{dt} \stackrel{\text{FT}}{\longleftrightarrow} j\omega X(j\omega)$$

Differentiation in Frequency

If

$$F[x(t)] = X(j\omega)$$

then

$$F[tx(t)] = j\frac{d}{d\omega}X(j\omega)$$

Time Integration

If

$$F[x(t)] = X(j\omega)$$

then

$$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(j\omega) + \pi X(0)\, \delta(\omega)$$

Time Scaling

If

$$F[x(t)] = X(j\omega)$$

then

$$F[x(at)] = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

For time reversal,

$$\boxed{F[x(-t)] = X(-j\omega)}$$

Frequency Shifting

If

$$F[x(t)] = X(j\omega)$$

then

$$F[x(t)e^{j\omega_0 t}] = X[j(\omega - \omega_0)]$$

Duality

If

$$F[x(t)] = X(j\omega)$$

then

$$F[X(t)] = 2\pi x(j\omega)$$

The Convolution

Let

$$y(t) = x(t) * h(t)$$

$$F[y(t)] = Y(j\omega) = X(j\omega)H(j\omega)$$

Parseval’s Theorem

According to Parseval’s theorem, the total energy in a signal is obtained by integrating the energy per unit frequency $\frac{|X(j\omega)|^2}{2\pi}$.

$$\boxed{E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega}$$

Signal	Fourier transform
1. $\delta(t)$	1
2. $u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
3. $\delta(t - t_0)$	$e^{-j\omega t_0}$
4. $te^{-at}u(t)$	$\frac{1}{(a + j\omega)^2}$
5. $u(-t)$	$\pi\delta(\omega) - \frac{1}{j\omega}$
6. $e^{at}u(-t)$	$\frac{1}{(a - j\omega)}$
7. $e^{-a\|t\|}$	$\frac{2a}{a^2 + \omega^2}$
8. $\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
9. $\sin\omega_0 t$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
10. $\frac{1}{(a^2 + t^2)}$	$e^{-a\|\omega\|}$
11. $\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
12. 1; for all t	$2\pi\,\delta(\omega)$

Table 6.1 Fourier transform properties

Property	Time signal $x(t)$	Fourier transform $X(j\omega)$
1. Linearity	$x(t) = A\,x_1(t) + B\,x_2(t)$	$X(j\omega) = A\,X_1(j\omega) + B\,X_2(j\omega)$
2. Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
3. Conjugation	$x^*(t)$	$X^*(-j\omega)$
4. Differentiation in time	$\dfrac{d^n x(t)}{dt^n}$	$(j\omega)^n X(j\omega)$
5. Differentiation in frequency	$tx(t)$	$j\dfrac{d}{d\omega}X(j\omega)$
6. Time integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\dfrac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
7. Time scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert}X\left(j\dfrac{\omega}{a}\right)$
8. Time reversal	$x(-t)$	$X(-j\omega)$
9. Frequency shifting	$x(t)e^{j\omega_0 t}$	$X[j(\omega - \omega_0)]$
10. Duality	$X(t)$	$2\pi x(j\omega)$
11. Time convolution	$x(t) * h(t)$	$X(j\omega)H(j\omega)$
12. Parseval's theorem	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt$	$E = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\,d\omega$

Find the Fourier transform

$$x(t) = e^{j\omega_0 t} = 1e^{j\omega_0 t}$$

Let $y(t) = 1$.

$$Y(j\omega) = 2\pi\delta(\omega)$$

By using the frequency shifting property, we get

$$\boxed{X(j\omega) = 2\pi\delta(\omega - \omega_0)}$$

$$\boldsymbol{x(t) = e^{-j\omega_0 t}}$$

$$\begin{aligned} x(t) &= e^{-j\omega_0 t} \\ &= e^{-j\omega_0 t} 1 \end{aligned}$$

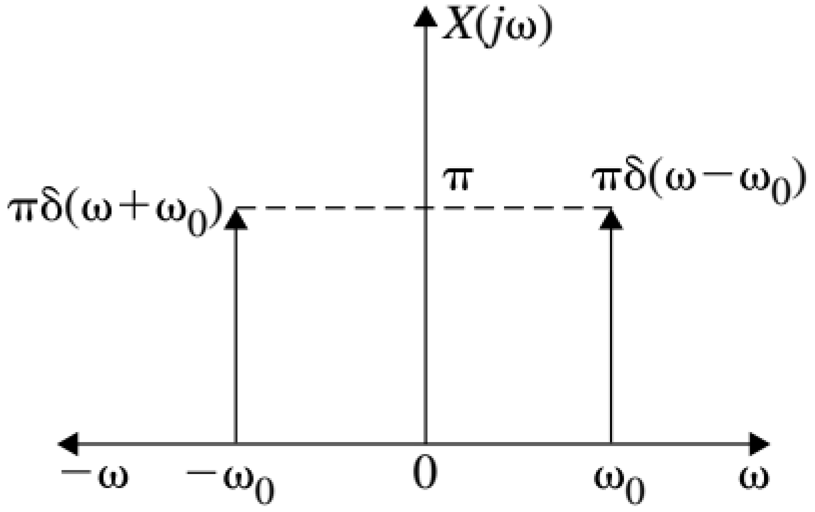
Since $1 \stackrel{\text{FT}}{\longleftrightarrow} 2\pi\delta(\omega)$, by using the frequency shifting property we get

$$\boxed{X(j\omega) = 2\pi\delta(\omega + \omega_0)}$$

$$x(t) = \cos(\omega_0 t)$$

$$\begin{aligned} x(t) &= \cos(\omega_0 t) \\ &= \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] \end{aligned}$$

$$\boxed{X(j\omega) = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]}$$

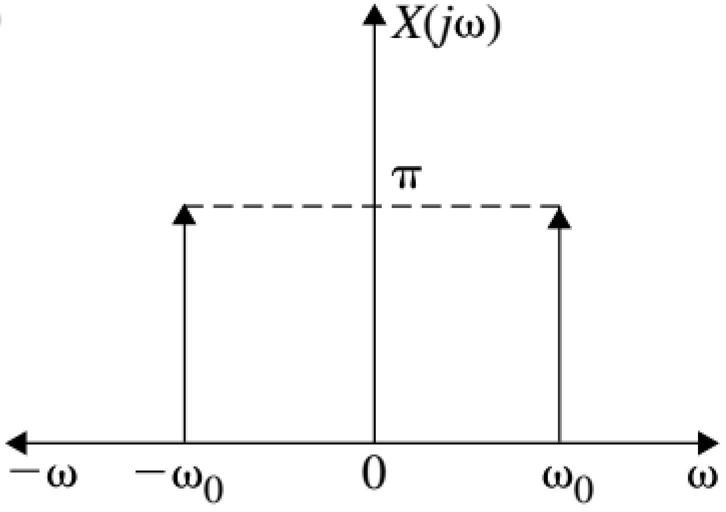


$$\boldsymbol{x(t) = \sin \omega_0 t}$$

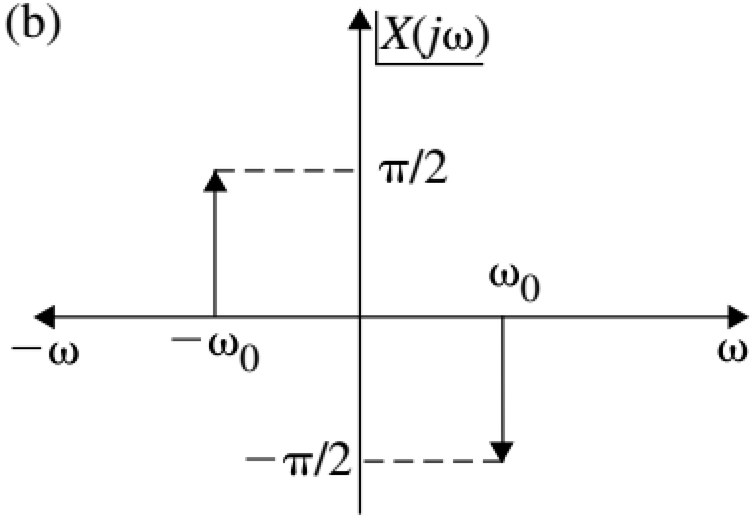
$$x(t) = \sin \omega_0 t$$
$$= \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$\boxed{X(j\omega) = -j\pi\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]}$$

(a)



(b)



Fourier Transform Using Differentiation and Integration Properties

$$x(t) \xleftrightarrow{\text{FT}} X(j\omega)$$

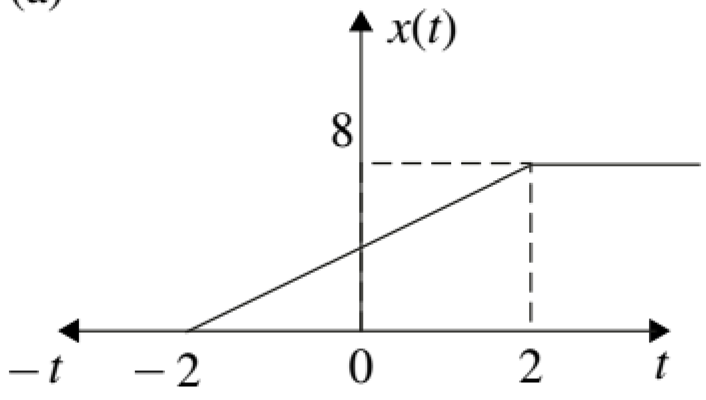
$$\frac{dx(t)}{dt} \xleftrightarrow{\text{FT}} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\text{FT}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

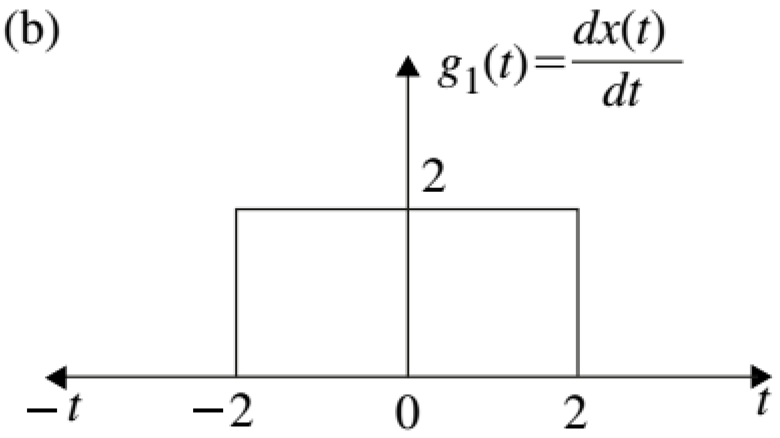
Find the CTFT for the following signal $x(t)$:

$$x(t) = \begin{cases} 2t + 4 & -2 \le t < 2 \\ 8 & 2 \le t < \infty \\ 0 & \text{otherwise} \end{cases}$$

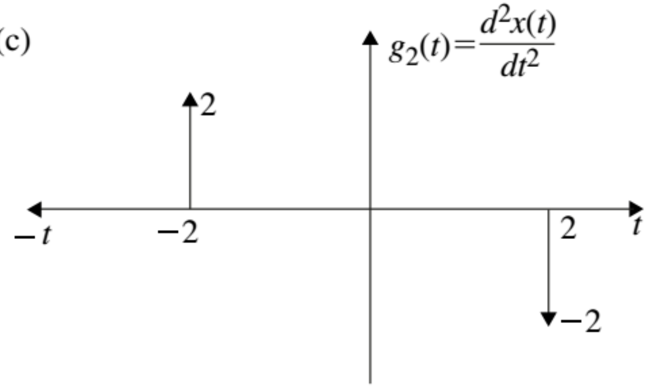
(a)



(b)



(c)



- Differentiate to get the simplest FT.
- Find FT of the nth derivative
- Divide by $j\omega$ n times

$$
\begin{aligned}
G_2(j\omega) &= [2e^{j2\omega} - 2e^{-j2\omega}] \\
&= 2\frac{[e^{j2\omega} - e^{-j2\omega}]2j}{2j} \\
&= j4\sin 2\omega \\
G_2(0) &= 0
\end{aligned}
$$

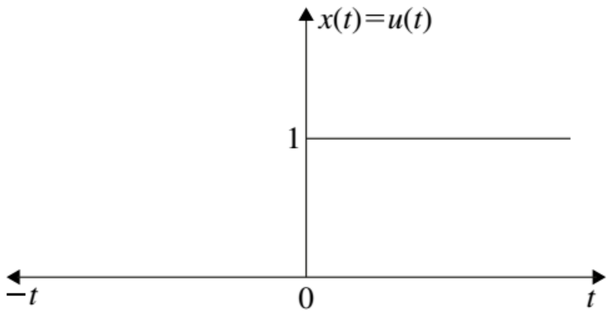
$$
\begin{aligned}
G_1(j\omega) &= \frac{G_2(j\omega)}{j\omega} + \pi\delta(\omega)G_2(0) \\
&= \frac{j4\sin 2\omega}{j\omega} \\
&= 8\left(\frac{\sin 2\omega}{2\omega}\right) \\
G_1(0) &= 8 \\
G_1(j\omega) &= 8\left(\frac{\sin 2\omega}{2\omega}\right) \qquad \left[\left.\frac{\sin 2\omega}{2\omega}\right|_{\omega=0} = 1\right]
\end{aligned}
$$

$$
X(j\omega) = \frac{G_1(j\omega)}{j\omega} + \pi\delta(\omega)G_1(0)
$$

$$
\boxed{X(j\omega) = \frac{4\sin 2\omega}{j\omega^2} + 8\pi\delta(\omega)}
$$

Find the FT of the step function $u(t)$ using the integration property of FT.

$$\delta(t) = \frac{du(t)}{dt}$$
$$du(t) = \delta(t)dt$$



Substituting $x(t) = u(t)$ and $\delta(t) = g(t)$

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

Taking FT on both sides, we get

$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega)$$

$$g(t) = \delta(t) \xleftrightarrow{\text{FT}} G(j\omega) = 1$$
$$G(0) = 1$$

Substituting the above in $X(j\omega)$, we get

$$\boxed{X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)}$$

For the following signals, determine the FT using FT properties.

$$x(t) = 5 \sin 10t$$

$y(t) = x(t - 3)$

$$X(j\omega) = j5\pi[\delta(\omega + 10) - \delta(\omega - 10)]$$
$$y(t) = x(t - 3)$$

FT of $y(t)$ is obtained using the time shifting property (right shift) as

$$\begin{aligned} Y(j\omega) &= X(j\omega)e^{-j3\omega} \\ &= j5\pi[\delta(\omega + 10) - \delta(\omega - 10)]e^{-j3\omega} \\ &= j5\pi[\delta(\omega + 10)e^{-j3\omega} - \delta(\omega - 10)e^{-j3\omega}] \end{aligned}$$

Using the property

$$X(j\omega)\delta(\omega - \omega_0) = X(j\omega_0)\delta(\omega - \omega_0)$$

we get

$$\boxed{Y(j\omega) = j5\pi[\delta(\omega + 10)e^{j30} - \delta(\omega - 10)e^{-j30}]}$$

Consider the following CT signal.

$$x(t) = 4\cos 3t$$

Determine the FT of the following signal

$y(t) = x(2-t) + x(-2-t)$

$$X(j\omega) = 4\pi[\delta(\omega+3) + \delta(\omega-3)]$$
$$y(t) = x(2-t) + x(-2-t)$$

$$\boxed{Y(j\omega) = 8\pi\cos 2\omega\,[\delta(\omega+3) + \delta(\omega-3)]}$$

$$\begin{aligned}
x(2-t) &\overset{\text{FT}}{\longleftrightarrow} X(-j\omega)e^{-j2\omega} \\
x(-2-t) &\overset{\text{FT}}{\longleftrightarrow} X(-j\omega)e^{j2\omega} \\
x(2-t) + x(-t-2) &\overset{\text{FT}}{\longleftrightarrow} X(-j\omega)[e^{j2\omega} + e^{-j2\omega}] \\
&= X(-j\omega)2\cos 2\omega \\
X(-j\omega) &= 4\pi[\delta(-\omega+3) + \delta(-\omega-3)] \\
Y(j\omega) &= X(-j\omega)2\cos 2\omega
\end{aligned}$$

$y(t) = \frac{d^2}{dt^2} x(t-2)$

$$x(t-2) \overset{\text{FT}}{\longleftrightarrow} 4\pi[\delta(\omega+3) + \delta(\omega-3)]e^{-j2\omega}$$
$$= 4\pi[\delta(\omega+3)e^{j6} + \delta(\omega-3)e^{-j6}]$$

Using differentiation property

$$\frac{d^2x}{dt^2} \overset{\text{FT}}{\longleftrightarrow} (j\omega)^2 X(j\omega)$$

we get

$$Y(j\omega) = 4(j\omega)^2\pi[\delta(\omega+3)e^{j6} + \delta(\omega-3)e^{-j6}]$$
$$= 4\pi[-\delta(\omega+3)9e^{j6} - \delta(\omega-3)9e^{-j6}]$$

$$\boxed{Y(j\omega) = -36\pi\left[\delta(\omega+3)\, e^{j6} + \delta(\omega-3)\, e^{-j6}\right]}$$

A signal has the following FT:

$$X(j\omega) = \frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3}$$

Find the FT of $x(-2t+1)$.

$$X(j\omega) = \frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3}$$

By using the time reversal property, the FT of $x(-t) = X(-j\omega)$ is obtained as

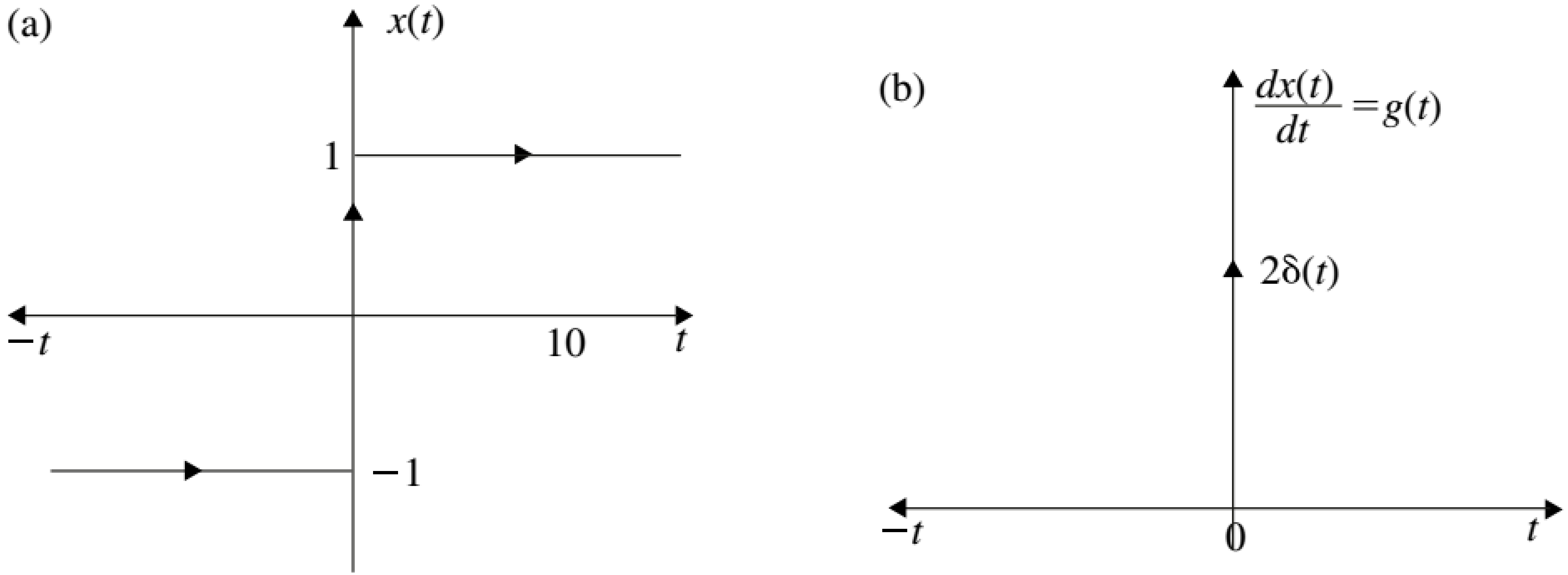
$$x(-t) \overset{FT}{\longleftrightarrow} \frac{\omega^2 - j4\omega + 2}{-\omega^2 - j4\omega + 3}$$

By using the time shifting (right shift) property, we get

$$x(-t+1) \overset{\text{FT}}{\longleftrightarrow} \left(\frac{\omega^2 - j4\omega + 2}{-\omega^2 - j4\omega + 3}\right) e^{-j\omega}$$

$$\boxed{x(-2t+1) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{2} e^{-j\omega/2} \frac{\left[\frac{\omega^2}{4} - j2\omega + 2\right]}{\left[-\frac{\omega^2}{4} - j2\omega + 2\right]}}$$

By using differentiation and integration property of FT, determine the FT of $x(t) = \text{sgn}(t)$.



$$\frac{dx(t)}{dt} = 2\delta(t) \stackrel{\text{FT}}{\longleftrightarrow} 2$$

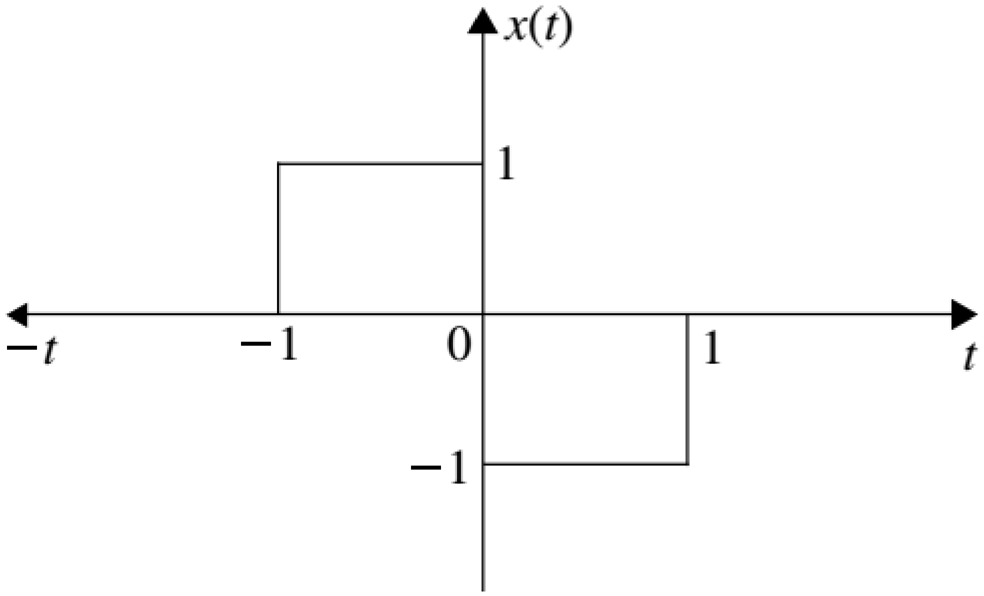
Using the integration property, we get

$$\begin{aligned} X(j\omega) &= \frac{1}{j\omega} FT\left[\frac{dx(t)}{dt}\right] + \pi\delta(\omega)G(0) \\ &= \frac{2}{j\omega} \end{aligned}$$

Since the area under the impulse is zero, the initial condition $G(0) = 0$.

$$\boxed{X(j\omega) = \frac{2}{j\omega}}$$

Find the Fourier transform of the signal



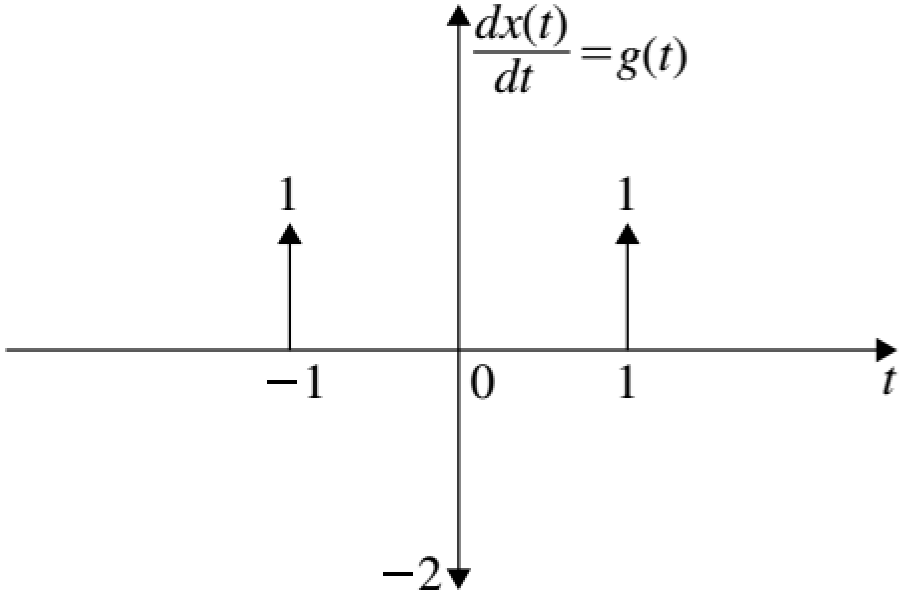
$$x(t) = \begin{cases} 1 & -1 \leq t \leq 0 \\ -1 & 0 \leq t \leq 1 \end{cases}$$

$$
\begin{aligned}
X(j\omega) &= \int_{-1}^{0} e^{-j\omega t}\,dt - \int_{0}^{1} e^{-j\omega t}\,dt \\
&= \frac{-1}{j\omega}\left\{\left[e^{-j\omega t}\right]_{-1}^{0} - \left[e^{-j\omega t}\right]_{0}^{1}\right\} \\
&= \frac{-1}{j\omega}\left[1 - e^{j\omega} - e^{-j\omega} + 1\right]
\end{aligned}
$$

$$
\boxed{X(j\omega) = \frac{2}{j\omega}[\cos\omega - 1]}
$$

$$G(j\omega) = F\left[\frac{dx(t)}{dt}\right] = \left[e^{j\omega} - 2 + e^{-j\omega}\right] = 2[\cos \omega$$

$$G(0) = 2[1-1] = 0$$



Using the time integration property, we get

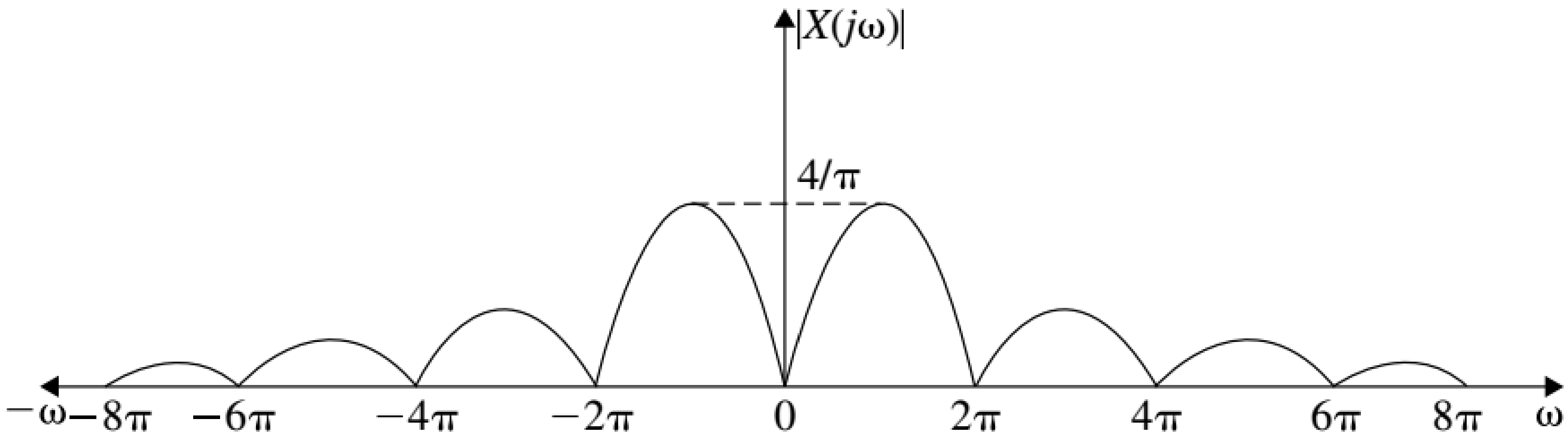
$$F[x(t)] = X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega) = \frac{2}{j\omega}[\cos\omega - 1] + \pi G(0)\delta(\omega)$$

$$\boxed{X(j\omega) = \frac{2}{j\omega}[\cos\omega - 1]}$$

To Plot the Magnitude Spectrum

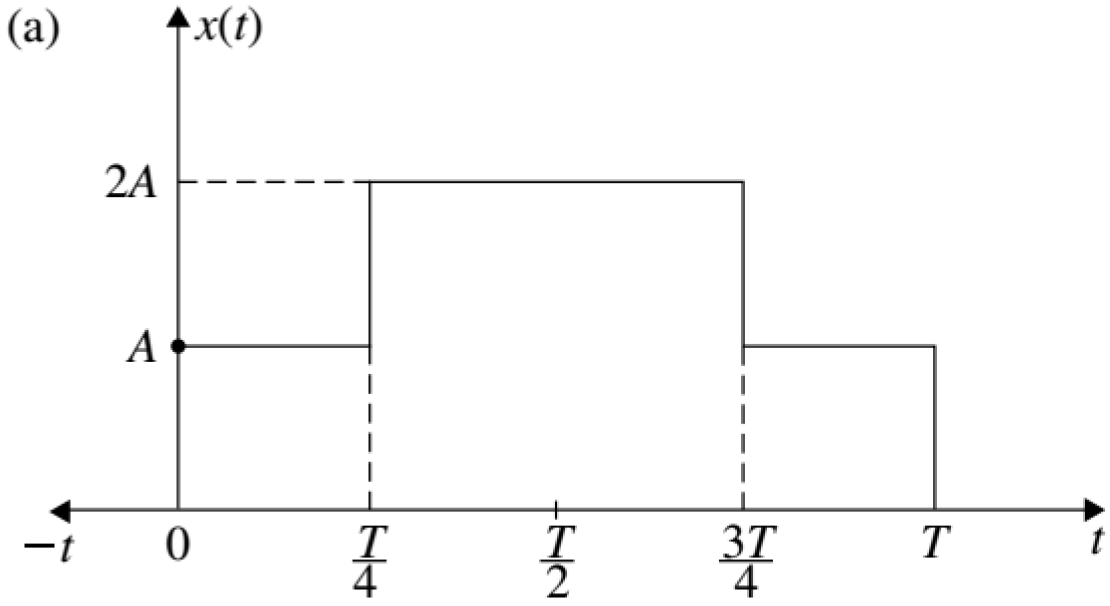
$$
\begin{aligned}
|X(j\omega)| &= \frac{2}{\omega}[\cos\omega - 1] \\
&= \frac{2}{\omega}\left[\cos^2\frac{\omega}{2} - \sin^2\frac{\omega}{2} - 1\right] \\
&= \frac{-4}{\omega}\sin^2\omega/2 \\
&= -\omega\left[\frac{\sin\omega/2}{\frac{\omega}{2}}\right]^2
\end{aligned}
$$

$$
\boxed{|X(j\omega)| = \left|\omega\,\mathrm{sinc}^2\frac{\omega}{2}\right|}
$$

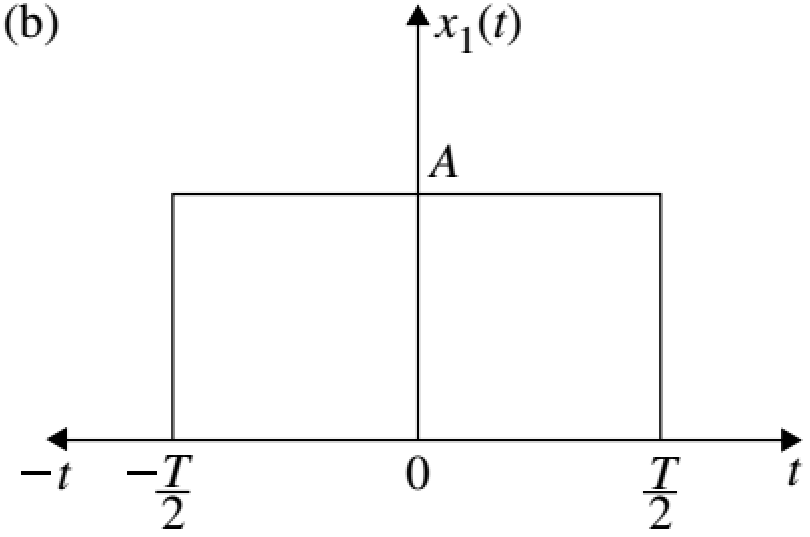


|X(jω)|
4/π
−ω
−8π
−6π
−4π
−2π
0
2π
4π
6π
8π
ω

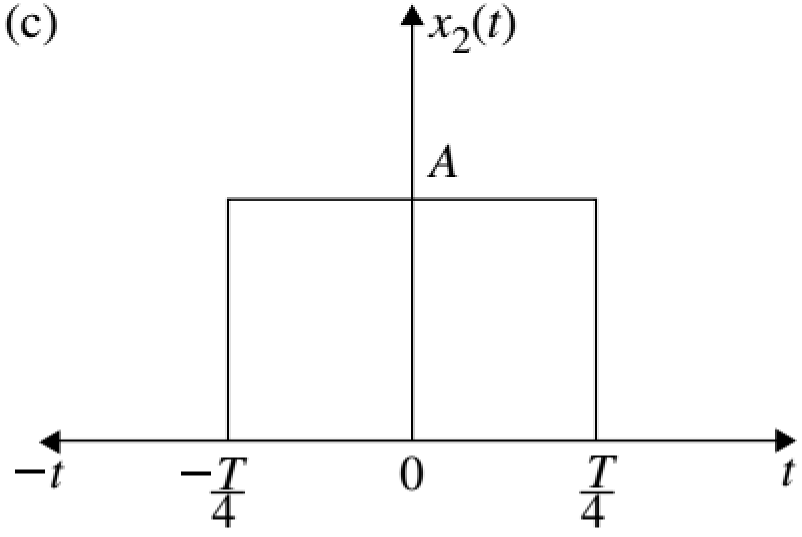
Using Fourier transform properties, find the Fourier transform of the signal shown by using
(a) Time shifting and (b) Differentiation and integration.



(b)



(c)



Method 1: Time Shifting Property

$$x(t) = A\left[x_1\left(t - \frac{T}{2}\right) + x_2\left(t - \frac{T}{2}\right)\right]$$

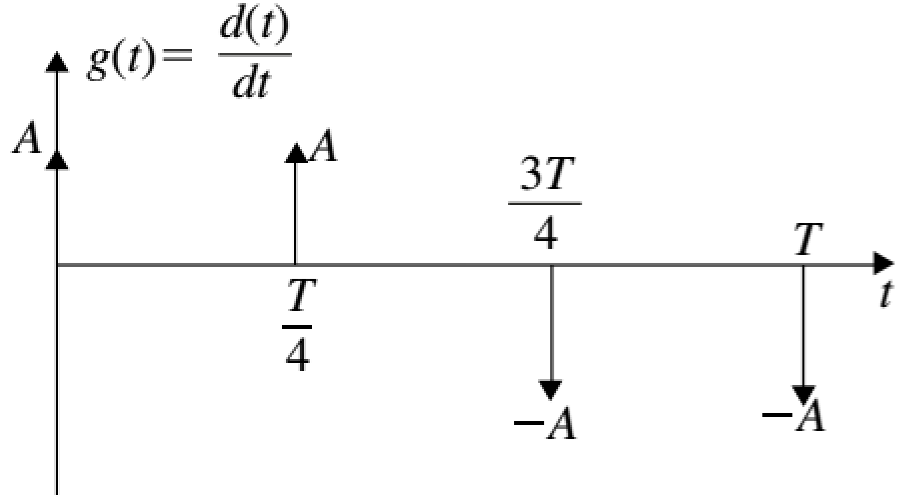
$$X_1(j\omega) = AT\operatorname{sinc}\frac{\omega T}{2}$$

$$X_2(j\omega) = \frac{1}{2}AT\operatorname{sinc}\frac{\omega T}{4}$$

$$X(j\omega) = [X_1(j\omega) + X_2(j\omega)]e^{-j\frac{\omega T}{2}}$$

$$\boxed{X(j\omega) = AT\left[\operatorname{sinc}\frac{\omega T}{2} + \frac{1}{2}\operatorname{sinc}\frac{\omega T}{4}\right]e^{-j\frac{\omega T}{2}}}$$

Method 2: Using Differentiation and Integration Properties



$$g(t) = \frac{dx}{dt}$$
$$= A\delta(t) + A\delta\left(t - \frac{T}{4}\right) - A\delta\left(t - \frac{3T}{4}\right) - A\delta(t - T)$$

Taking FT on both sides, we get

$$
\begin{aligned}
G(j\omega) &= A[1 + e^{-j\omega(T/4)} - e^{-j\omega(3T/4)} - e^{-j\omega T}] \\
G(0) &= A[1 + 1 - 1 - 1] \\
&= 0 \qquad \text{Note: If} x(t) \text{ is finite for } t \to \infty, G(0) = 0 \\
G(j\omega) &= Ae^{-j\omega(T/2)}[(e^{j\omega(T/2)} - e^{-j\omega(T/2)}) + (e^{j\omega(T/4)} - e^{-j\omega(T/4)})] \\
&= 2Aj\left[\sin\frac{\omega T}{2} + \sin\frac{\omega T}{4}\right]e^{-j\omega(T/2)}
\end{aligned}
$$

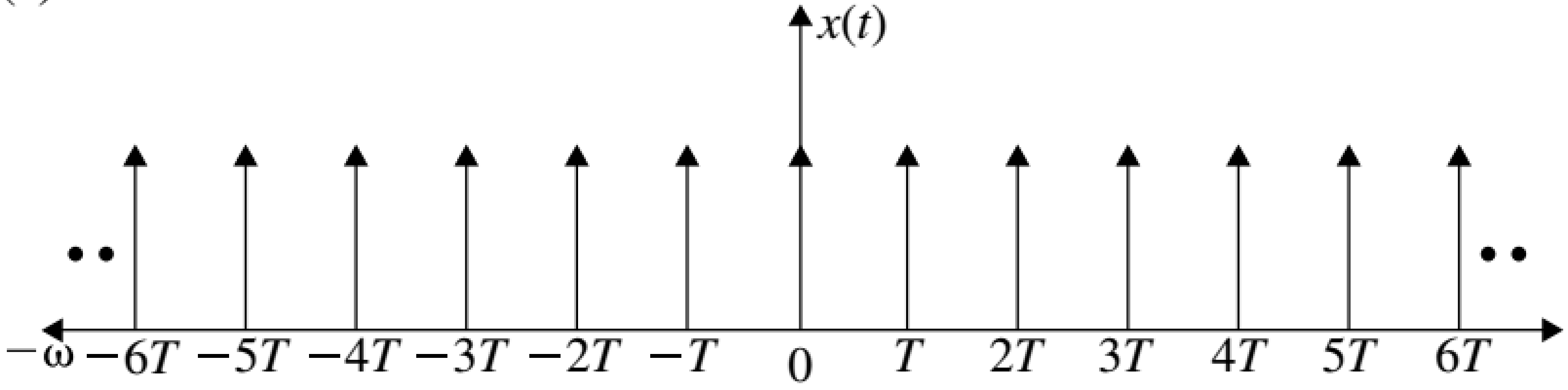
The FT of $x(t)$ is obtained by integrating $G(j\omega)$. Thus

$$
\begin{aligned}
X(j\omega) &= \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega) \\
&= \frac{2A}{\omega}\left[\sin\frac{\omega T}{2} + \sin\frac{\omega T}{4}\right]e^{-j\omega(T/2)} + 0 \\
&= AT\left[\frac{\sin\frac{\omega T}{2}}{\left(\frac{\omega T}{2}\right)} + \frac{1}{2}\frac{\sin\frac{\omega T}{4}}{\left(\frac{\omega T}{4}\right)}\right]e^{-j\omega(T/2)} \\
&= AT\left[\operatorname{sinc}\frac{\omega T}{2} + \frac{1}{2}\operatorname{sinc}\frac{\omega T}{4}\right]e^{-j\omega(T/2)}
\end{aligned}
$$

Find the Fourier transform of the impulse train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

(a)



where T is the periodicity. The Fourier series coefficients are determined as

$$\begin{aligned} D_n &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jn\omega_0 t}\,dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-0}\,dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)dt \\ &= \frac{1}{T} \end{aligned}$$

For a periodic signal

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt}$$

where

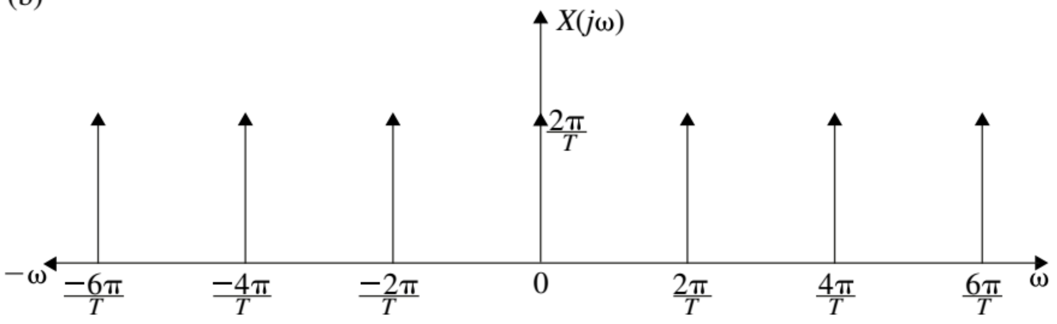
$$\omega_0 = \frac{2\pi}{T}$$

and

$$X(j\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0)$$

$$X(j\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right)$$

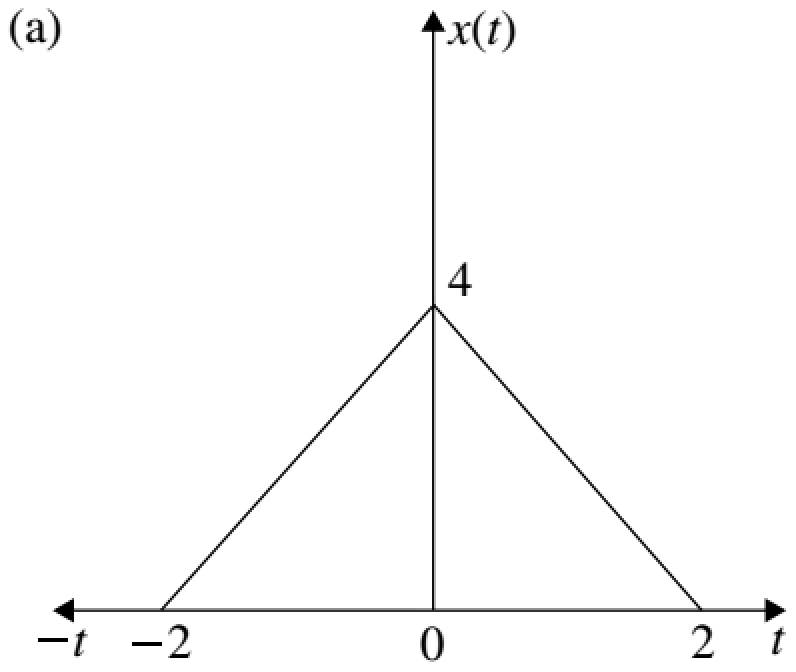
(b)



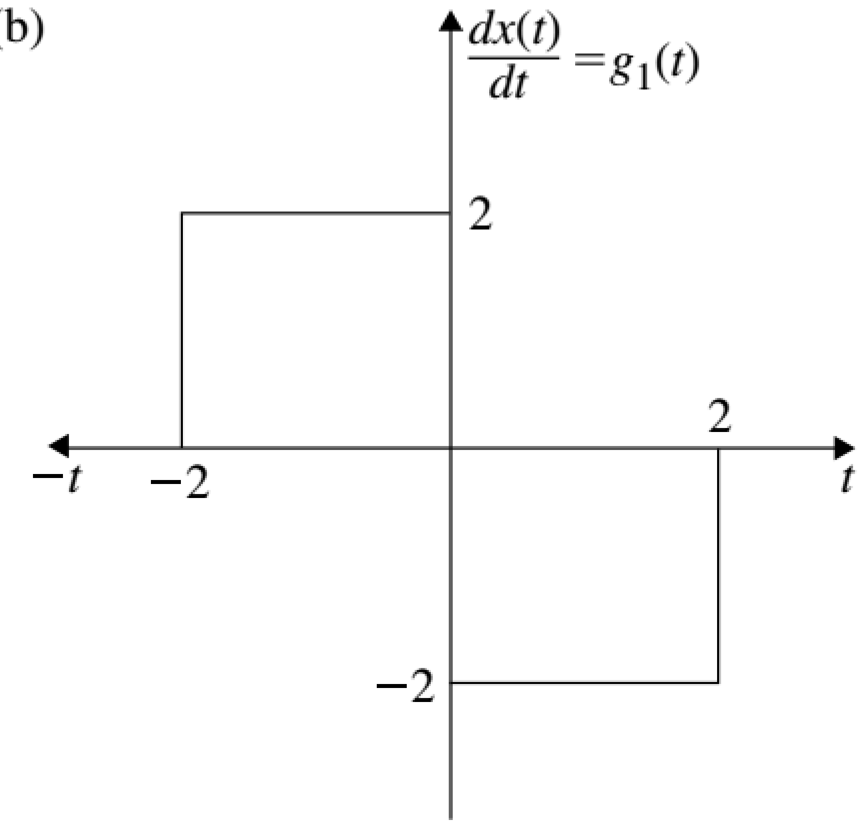
find the Fourier transform using differentiation and integration properties.

$$x(t) = \begin{cases} 2t + 4 & -2 \le t < 0 \\ 4 - 2t & 0 \le t \le 2 \end{cases}$$

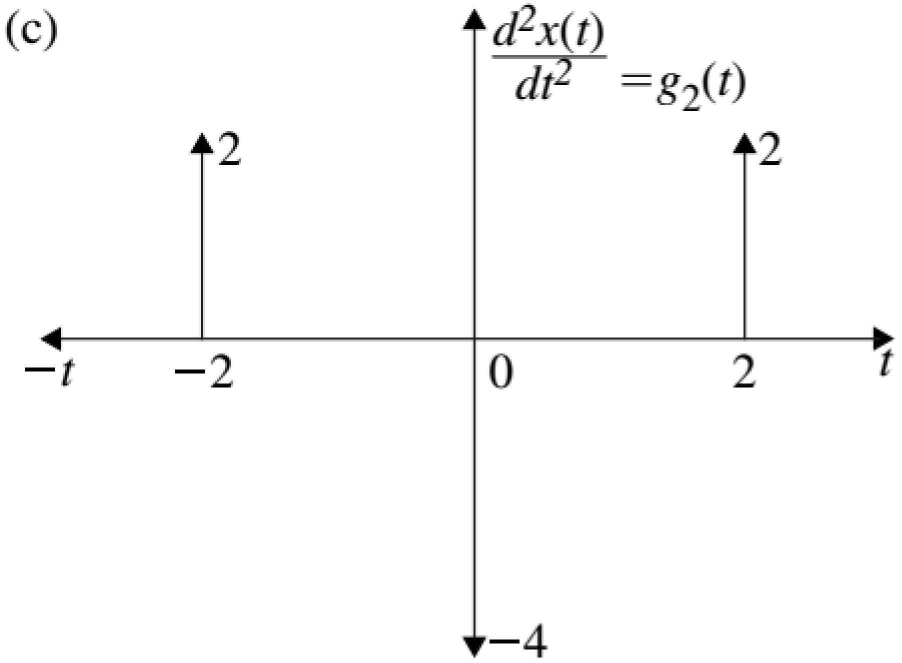
$$\frac{dx(t)}{dt} = \begin{cases} 2 & -2 \le t < 0 \\ -2 & 0 \le t \le 2 \end{cases}$$



(b)



(c)



$$\frac{d^2x(t)}{dt^2} = \begin{cases} 2\delta(t+2) & t=-2 \\ -4 & t=0 \\ 2\,\delta(t-2) & t=2 \end{cases}$$

$$F\left[\frac{d^2x(t)}{dt^2}\right] = G_2(j\omega) = 2e^{j2\omega} - 4 + 2e^{-j2\omega}$$
$$= 4[\cos 2\omega - 1]$$
$$G_2(j\omega) = -8\sin^2\omega$$
$$G_2(0) = 0$$

$X(j\omega)$ is obtained by dividing $G_1(j\omega)$ by $(j\omega)^2$ and adding initial condition

$$X(j\omega) = \frac{G_2(j\omega)}{(j\omega)^2} + \pi G_2(0)\delta(\omega)$$
$$= \frac{-8}{(j\omega)^2}\sin^2\omega$$
$$= 8\left[\frac{\sin\omega}{\omega}\right]^2$$

$$\boxed{X(j\omega) = 8\text{sinc}^2\,\omega}$$

Find the Fourier transform of

$$x(t) = \frac{2a}{a^2 + t^2}$$

using the duality property of FT.

the FT of $x(t) = e^{-a|t|}$ is obtained as

$$x(t) = e^{-a|t|} \stackrel{\text{FT}}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

By the application of inverse Fourier transform, we get

$$e^{-a|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2} e^{j\omega t}\, d\omega$$

$$2\pi e^{-a|t|} = \int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2} e^{j\omega t}\, d\omega$$

Replacing t by $-t$ in the above equation, we get

$$2\pi e^{-a|t|} = \int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2} e^{-j\omega t}\, d\omega$$

Interchanging t and ω in the above equation, we get

$$2\pi e^{-a|\omega|} = \int_{-\infty}^{\infty} \frac{2a}{(a^2+t^2)} e^{-j\omega t}\, dt$$

The right-hand side of the above equation is nothing but the FT of $\frac{2a}{a^2+t^2}$.

$$2\pi e^{-a|\omega|} = F\left[\frac{2a}{(a^2+t^2)}\right]$$

$$\boxed{\left[\frac{2a}{(a^2+t^2)}\right] \xleftrightarrow{\text{FT}} 2\pi e^{-a|\omega|}}$$

Method 2

The duality property of $X(t) = 2\pi x(-\omega)$.

$$e^{-a|t|} \overset{\text{FT}}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

$$X(t) = \frac{2a}{a^2 + t^2}$$

$$x(-\omega) = e^{-a|\omega|}$$

$$X(t) \overset{\text{FT}}{\longleftrightarrow} 2\pi x(-\omega)$$

$$\boxed{\frac{2a}{a^2 + t^2} \overset{\text{FT}}{\longleftrightarrow} 2\pi e^{-a|\omega|}}$$

- See example 6.23 for more applications

Using the properties of continuous time Fourier transform, determine the time domain signal $x(t)$.

If the frequency domain signal is described as given below.

$$X(j\omega) = j\frac{d}{d\omega}\left[\frac{e^{j2\omega}}{(1+\frac{j\omega}{3})}\right]$$

First, the time scaling property is applied. Let

$$X_1(j\omega) = \frac{1}{1+j\omega}$$

$$x_1(t) = e^{-t}u(t)$$

$$F\left[x_1[3t]\right] = 3e^{-3t}u(3t)$$

$$F\left[3e^{-3t}u(3t)\right] = \frac{1}{\left[1+\frac{j\omega}{3}\right]}$$

$$F^{-1}\left[\frac{1}{\left(1+\frac{j\omega}{3}\right)}\right] = 3e^{-3t}u(t) \quad [\because u(t) = u(3t)]$$

According to the time shifting property,

$$e^{j2\omega}Y(j\omega) = y(t+2)$$

$$F^{-1}\left[\frac{e^{j2\omega}}{\left(1+\frac{j\omega}{3}\right)}\right] = 3e^{-3(t+2)}u(t+2)$$

According to differentiating property,

$$j\frac{d}{d\omega}X(j\omega) = tx(t)$$

Applying the above property, we have

$$F^{-1}\left[j\frac{d}{d\omega}\frac{e^{j2\omega}}{\left(1+\frac{j\omega}{3}\right)}\right]=3te^{-3(t+2)}u(t+2)$$

$$\therefore\quad X(j\omega)=\frac{jd}{d\omega}\left[\frac{e^{j2\omega}}{\left(1+\frac{j\omega}{3}\right)}\right]$$

$$\boxed{x(t)=3te^{-3(t+2)}u(t+2)}$$

Find the inverse Fourier transform of the following functions:

$\boldsymbol{X(j\omega) = \delta(\omega - \omega_0)}$

The IFT of $\delta(\omega) = \frac{1}{2\pi}$. $\delta(\omega)$ is frequency-shifted by ω_0.

$$F^{-1}\left[X(j\omega)\right] = e^{j\omega_0 t}\frac{1}{2\pi}$$

$$\boxed{F^{-1}\left[\delta(\omega - \omega_0)\right] = \frac{1}{2\pi}e^{j\omega_0 t}}$$

$$\boldsymbol{X(j\omega) = \frac{j\omega}{(2+j\omega)^2}}$$

$$F\left[e^{-2t}\right] = \frac{1}{(2+j\omega)}$$

By applying

$$F\left[te^{-2t}\right] = \frac{d}{d\omega}\frac{1}{(2+j\omega)}$$

(Applying frequency differentiation)

$$F\left[te^{-2t}\right] = \frac{1}{(2+j\omega)^2}$$

$$\therefore \quad F^{-1}\left[\frac{1}{(2+j\omega)^2}\right] = te^{-2t}$$

By applying time differentiation, namely

$$\frac{dx(t)}{dt} = j\omega X(j\omega)$$

$$\boxed{F^{-1}\left[\frac{j\omega}{(2+j\omega^2)}\right] = \frac{d}{dt}\left(te^{-2t}\right)}$$

$\boldsymbol{X(j\omega) = \frac{6}{(\omega^2+9)}}$

$$X(j\omega) = \frac{-6}{(j\omega+3)(j\omega-3)}$$
$$= \frac{A_1}{j\omega+3} + \frac{A_2}{j\omega-3}$$
$$-6 = A_1(j\omega-3) + A_2(j\omega+3)$$

Let $j\omega = -3$

$$A_1 = 1$$

Let $j\omega = 3$

$$A_2 = -1$$
$$X(j\omega) = \frac{1}{j\omega+3} - \frac{1}{j\omega-3}$$
$$x(t) = F^{-1}\left[X(j\omega)\right] = e^{-3t}u(t) + e^{3t}u(-t)$$

$$\boldsymbol{X(j\omega) = \frac{(j\omega+2)}{[(j\omega)^2+4j\omega+3]}}$$

$$
\begin{aligned}
X(j\omega) &= \frac{(j\omega+2)}{(j\omega+1)(j\omega+3)} \\
&= \frac{A_1}{(j\omega+1)} + \frac{A_2}{(j\omega+3)} \\
(j\omega+2) &= A_1(j\omega+3) + A_2(j\omega+1)
\end{aligned}
$$

Let $j\omega = -1$,

$$
\begin{aligned}
1 &= 2A_1 \\
A_1 &= \frac{1}{2}
\end{aligned}
$$

Let $j\omega = -3, \quad A_2 = \frac{1}{2}$

$$X(j\omega) = \frac{1}{2}\left[\frac{1}{j\omega+1} + \frac{1}{j\omega+3}\right]$$

$$\boxed{x(t) = \frac{1}{2}\left[e^{-t} + e^{-3t}\right]u(t)}$$

$$\boldsymbol{X(j\omega) = \frac{(j\omega+1)}{(j\omega+2)^2(j\omega+3)}}$$

$$X(j\omega) = \frac{A_1}{(j\omega+2)^2} + \frac{A_2}{(j\omega+2)} + \frac{A_3}{(j\omega+3)}$$

$$(j\omega+1) = A_1(j\omega+3) + A_2(j\omega+2)(j\omega+3) + A_3(j\omega+2)^2$$

Let $j\omega = -2$;

$$-1 = A_1$$

Let $j\omega = -3$;

$$-2 = A_3$$

$$(j\omega+1) = A_1(j\omega+3) + A_2\left[(j\omega)^2 + 5j\omega + 6\right] + A_3\left[(j\omega)^2 + 4j\omega + 4\right]$$

Compare the coefficients of $j\omega$ on both sides,

$$\begin{aligned} 1 &= A_1 + 5A_2 + 4A_3 \\ &= -1 + 5A_2 - 8 \\ A_2 &= 2 \\ X(j\omega) &= \frac{-1}{(j\omega+2)^2} + \frac{2}{(j\omega+2)} - \frac{2}{(j\omega+3)} \\ x(t) &= F^{-1}[x(j\omega)] \end{aligned}$$

$$\boxed{x(t) = \left[-te^{-2t} + 2e^{-2t} - 2e^{-3t}\right] u(t)}$$

Consider a causal LTI system with frequency response,

$$H(j\omega) = \frac{1}{j\omega + 3}$$

For a particular input $x(t)$, this system is to produce the output

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

Determine $x(t)$.

$$
\begin{aligned}
y(t) &= e^{-3t}u(t) - e^{-4t}u(t) \\
Y(j\omega) &= \frac{1}{(j\omega+3)} - \frac{1}{(j\omega+4)} \\
&= \frac{1}{(j\omega+3)(j\omega+4)} \\
H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\
X(j\omega) &= \frac{Y(j\omega)}{H(j\omega)} \\
&= \frac{(j\omega+3)}{(j\omega+3)(j\omega+4)} \\
&= \frac{1}{(j\omega+4)} \\
x(t) &= F^{-1}X(j\omega) = e^{-4t}u(t)
\end{aligned}
$$

$$
\boxed{x(t) = e^{-4t}u(t)}
$$

Find the Fourier transform of the following signals using convolution theorem.

$$x(t) = e^{-2t}u(t) * e^{-5t}u(t)$$

$$X(j\omega) = F\left[e^{-2t}u(t)\right] F\left[e^{-5t}u(t)\right]$$

$$F\left[e^{-2t}u(t)\right] = \frac{1}{(j\omega + 2)}$$

$$F\left[e^{-5t}u(t)\right] = \frac{1}{(j\omega + 5)}$$

$$\boxed{X(j\omega) = \frac{1}{(j\omega+2)(j\omega+5)}}$$

$$X(j\omega) = \frac{1}{3}\left[\frac{1}{j\omega+2} - \frac{1}{(j\omega+5)}\right]$$

$$x(t) = F^{-1}[X(j\omega)] = \frac{1}{3}[e^{-2t}u(t) - e^{-5t}u(t)]$$

$$\boxed{x(t) = \frac{1}{3}\left[e^{-2t} - e^{-5t}\right]u(t)}$$

Consider the following signals $x_1(t)$ and $x_2(t)$. Find

$$y(t) = x_1(t) * x_2(t)$$

$$\mathbf{x_1(t) = e^{-2t}u(t) \text{ and } x_2(t) = e^{3t}u(-t)}$$

$$X_1(j\omega) = \frac{1}{(j\omega + 2)}$$

$$X_2(j\omega) = -\frac{1}{(j\omega - 3)}$$

$$x_1(t) * x_2(t) = X_1(j\omega)X_2(j\omega)$$

$$Y(j\omega) = \frac{1}{(j\omega+2)}\frac{(-1)}{(j\omega-3)}$$

$$Y(j\omega) = \frac{A_1}{(j\omega+2)} + \frac{A_2}{(j\omega-3)}$$

$$= \frac{1}{5}\left[\frac{1}{j\omega+2} - \frac{1}{j\omega-3}\right]$$

$$y(t) = F^{-1}[Y(j\omega)] = \frac{1}{5}\left[e^{-2t}u(t) + e^{3t}u(-t)\right]$$

$$\boxed{y(t) = \frac{1}{5}\left[e^{-2t}u(t) + e^{3t}u(-t)\right]}$$

the “Modulation” property which states that

$$x(t)\cos\omega_0 t \xleftrightarrow{\text{FT}} \frac{1}{2}[X(\omega-\omega_0)+X(\omega+\omega_0)]$$

$x(t) = e^{-at} \cos \omega_0 t \, u(t)$

$$
\begin{aligned}
\cos \omega_0 t &= \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] \\
X(j\omega) &= \int_0^{\infty} e^{-at} \cos \omega_0 t e^{-j\omega t}\, dt \\
&= \frac{1}{2}\int_0^{\infty} e^{-at} e^{j\omega_0 t} e^{-j\omega t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-at} e^{-j\omega_0 t} e^{-j\omega t}\, dt \\
&= \frac{1}{2}\int_0^{\infty} e^{-(a-j\omega_0+j\omega)t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-(a+j\omega_0+j\omega)t}\, dt \\
&= \frac{1}{2}\left[\frac{-1}{(a-j\omega_0+j\omega)} e^{-(a-j\omega_0+j\omega)t} - \frac{e^{-(a+j\omega_0+j\omega)t}}{(a+j\omega_0+j\omega)}\right]_0^{\infty} \\
&= \frac{1}{2}\left[\frac{1}{(a+j\omega)-j\omega_0} + \frac{1}{(a+j\omega)+j\omega_0}\right] \\
&= \frac{1}{2}\frac{[a+j\omega+j\omega_0+a+j\omega-j\omega_0]}{(a+j\omega)^2+\omega_0^2}
\end{aligned}
$$

$$
\boxed{X(j\omega) = \frac{(a+j\omega)}{(a+j\omega)^2+\omega_0^2}}
$$