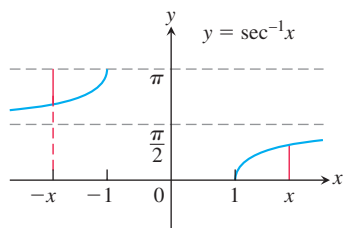


**46. Two derivations of the identity  $\sec^{-1}(-x) = \pi - \sec^{-1}x$**

- a. (*Geometric*) Here is a pictorial proof that  $\sec^{-1}(-x) = \pi - \sec^{-1}x$ . See if you can tell what is going on.



- b. (*Algebraic*) Derive the identity  $\sec^{-1}(-x) = \pi - \sec^{-1}x$  by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1}x \quad \text{Eq. (4), Section 1.6}$$

$$\sec^{-1}x = \cos^{-1}(1/x) \quad \text{Eq. (1)}$$

Which of the expressions in Exercises 47–50 are defined, and which are not? Give reasons for your answers.

47. a.  $\tan^{-1} 2$                       b.  $\cos^{-1} 2$   
 48. a.  $\csc^{-1}(1/2)$                 b.  $\csc^{-1} 2$   
 49. a.  $\sec^{-1} 0$                       b.  $\sin^{-1}\sqrt{2}$   
 50. a.  $\cot^{-1}(-1/2)$                 b.  $\cos^{-1}(-5)$   
 51. Use the identity

$$\csc^{-1}u = \frac{\pi}{2} - \sec^{-1}u$$

to derive the formula for the derivative of  $\csc^{-1}u$  in Table 3.1 from the formula for the derivative of  $\sec^{-1}u$ .

52. Derive the formula

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

for the derivative of  $y = \tan^{-1}x$  by differentiating both sides of the equivalent equation  $\tan y = x$ .

53. Use the Derivative Rule in Section 3.8, Theorem 3, to derive

$$\frac{d}{dx} \sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

54. Use the identity

$$\cot^{-1}u = \frac{\pi}{2} - \tan^{-1}u$$

to derive the formula for the derivative of  $\cot^{-1}u$  in Table 3.1 from the formula for the derivative of  $\tan^{-1}u$ .

55. What is special about the functions

$$f(x) = \sin^{-1} \frac{x-1}{x+1}, \quad x \geq 0, \quad \text{and} \quad g(x) = 2 \tan^{-1} \sqrt{x}?$$

Explain.

56. What is special about the functions

$$f(x) = \sin^{-1} \frac{1}{\sqrt{x^2+1}} \quad \text{and} \quad g(x) = \tan^{-1} \frac{1}{x}?$$

Explain.

- T** 57. Find the values of

a.  $\sec^{-1} 1.5$                       b.  $\csc^{-1}(-1.5)$                       c.  $\cot^{-1} 2$

- T** 58. Find the values of

a.  $\sec^{-1}(-3)$                       b.  $\csc^{-1} 1.7$                       c.  $\cot^{-1}(-2)$

- T** In Exercises 59–61, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

59. a.  $y = \tan^{-1}(\tan x)$                       b.  $y = \tan(\tan^{-1}x)$   
 60. a.  $y = \sin^{-1}(\sin x)$                       b.  $y = \sin(\sin^{-1}x)$   
 61. a.  $y = \cos^{-1}(\cos x)$                       b.  $y = \cos(\cos^{-1}x)$

- T** Use your graphing utility for Exercises 62–66.

62. Graph  $y = \sec(\sec^{-1}x) = \sec(\cos^{-1}(1/x))$ . Explain what you see.  
 63. **Newton's serpentine** Graph Newton's serpentine,  $y = 4x/(x^2+1)$ . Then graph  $y = 2 \sin(2 \tan^{-1}x)$  in the same graphing window. What do you see? Explain.  
 64. Graph the rational function  $y = (2-x^2)/x^2$ . Then graph  $y = \cos(2 \sec^{-1}x)$  in the same graphing window. What do you see? Explain.  
 65. Graph  $f(x) = \sin^{-1}x$  together with its first two derivatives. Comment on the behavior of  $f$  and the shape of its graph in relation to the signs and values of  $f'$  and  $f''$ .  
 66. Graph  $f(x) = \tan^{-1}x$  together with its first two derivatives. Comment on the behavior of  $f$  and the shape of its graph in relation to the signs and values of  $f'$  and  $f''$ .

## 3.10 Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a *related rates problem*.

### Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If  $V$  is the volume and  $r$  is the radius of the balloon at an instant of time, then

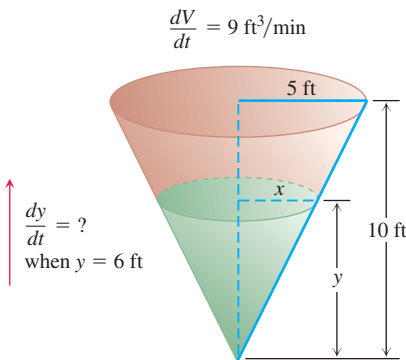
$$V = \frac{4}{3}\pi r^3.$$

Using the Chain Rule, we differentiate both sides with respect to  $t$  to find an equation relating the rates of change of  $V$  and  $r$ ,

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius  $r$  of the balloon and the rate  $dV/dt$  at which the volume is increasing at a given instant of time, then we can solve this last equation for  $dr/dt$  to find how fast the radius is increasing at that instant. Note that it is easier to directly measure the rate of increase of the volume (the rate at which air is being pumped into the balloon) than it is to measure the increase in the radius. The related rates equation allows us to calculate  $dr/dt$  from  $dV/dt$ .

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.



**FIGURE 3.44** The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

**EXAMPLE 1** Water runs into a conical tank at the rate of  $9 \text{ ft}^3/\text{min}$ . The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

**Solution** Figure 3.44 shows a partially filled conical tank. The variables in the problem are

$V$  = volume ( $\text{ft}^3$ ) of the water in the tank at time  $t$  (min)

$x$  = radius (ft) of the surface of the water at time  $t$

$y$  = depth (ft) of the water in the tank at time  $t$ .

We assume that  $V$ ,  $x$ , and  $y$  are differentiable functions of  $t$ . The constants are the dimensions of the tank. We are asked for  $dy/dt$  when

$$y = 6 \text{ ft} \quad \text{and} \quad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves  $x$  as well as  $V$  and  $y$ . Because no information is given about  $x$  and  $dx/dt$  at the time in question, we need to eliminate  $x$ . The similar triangles in Figure 3.44 give us a way to express  $x$  in terms of  $y$ :

$$\frac{x}{y} = \frac{5}{10} \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore, we find

$$V = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \frac{dy}{dt} = \frac{\pi}{4}y^2 \frac{dy}{dt}.$$

Finally, use  $y = 6$  and  $dV/dt = 9$  to solve for  $dy/dt$ .

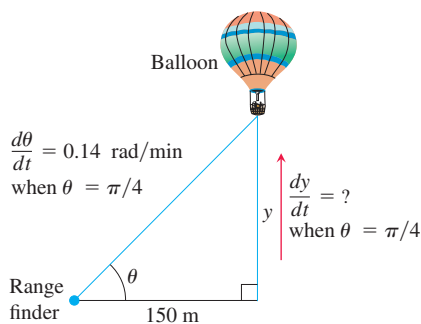
$$9 = \frac{\pi}{4}(6)^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min. ■

### Related Rates Problem Strategy

1. Draw a picture and name the variables and constants. Use  $t$  for time. Assume that all variables are differentiable functions of  $t$ .
2. Write down the numerical information (in terms of the symbols you have chosen).
3. Write down what you are asked to find (usually a rate, expressed as a derivative).
4. Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. Differentiate with respect to  $t$ . Then express the rate you want in terms of the rates and variables whose values you know.
6. Evaluate. Use known values to find the unknown rate.



**FIGURE 3.45** The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

**EXAMPLE 2** A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is  $\pi/4$ , the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

**Solution** We answer the question in the six strategy steps.

1. Draw a picture and name the variables and constants (Figure 3.45). The variables in the picture are
  - $\theta$  = the angle in radians the range finder makes with the ground.
  - $y$  = the height in meters of the balloon above the ground.

We let  $t$  represent time in minutes and assume that  $\theta$  and  $y$  are differentiable functions of  $t$ .

The one constant in the picture is the distance from the range finder to the liftoff point (150 m). There is no need to give it a special symbol.

2. Write down the additional numerical information.

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \quad \text{when} \quad \theta = \frac{\pi}{4}$$

3. Write down what we are to find. We want  $dy/dt$  when  $\theta = \pi/4$ .
4. Write an equation that relates the variables  $y$  and  $\theta$ .

$$\frac{y}{150} = \tan \theta \quad \text{or} \quad y = 150 \tan \theta$$

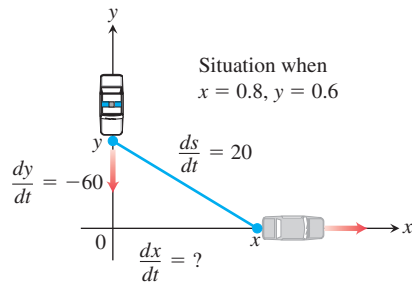
5. Differentiate with respect to  $t$  using the Chain Rule. The result tells how  $dy/dt$  (which we want) is related to  $d\theta/dt$  (which we know).

$$\frac{dy}{dt} = 150 (\sec^2 \theta) \frac{d\theta}{dt}$$

6. Evaluate with  $\theta = \pi/4$  and  $d\theta/dt = 0.14$  to find  $dy/dt$ .

$$\frac{dy}{dt} = 150(\sqrt{2})^2(0.14) = 42 \quad \sec \frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 42 m/min. ■



**FIGURE 3.46** The speed of the car is related to the speed of the police cruiser and the rate of change of the distance  $s$  between them (Example 3).

**EXAMPLE 3** A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

**Solution** We picture the car and cruiser in the coordinate plane, using the positive  $x$ -axis as the eastbound highway and the positive  $y$ -axis as the southbound highway (Figure 3.46). We let  $t$  represent time and set

$$\begin{aligned} x &= \text{position of car at time } t \\ y &= \text{position of cruiser at time } t \\ s &= \text{distance between car and cruiser at time } t. \end{aligned}$$

We assume that  $x$ ,  $y$ , and  $s$  are differentiable functions of  $t$ .

We want to find  $dx/dt$  when

$$x = 0.8 \text{ mi}, \quad y = 0.6 \text{ mi}, \quad \frac{dy}{dt} = -60 \text{ mph}, \quad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that  $dy/dt$  is negative because  $y$  is decreasing.

We differentiate the distance equation between the car and the cruiser,

$$s^2 = x^2 + y^2$$

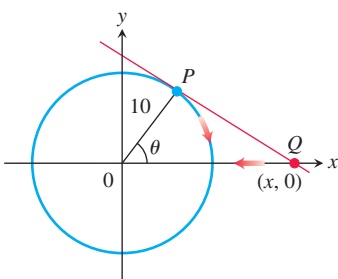
(we could also use  $s = \sqrt{x^2 + y^2}$ ), and obtain

$$\begin{aligned} 2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s} \left( x \frac{dx}{dt} + y \frac{dy}{dt} \right) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \left( x \frac{dx}{dt} + y \frac{dy}{dt} \right). \end{aligned}$$

Finally, we use  $x = 0.8$ ,  $y = 0.6$ ,  $dy/dt = -60$ ,  $ds/dt = 20$ , and solve for  $dx/dt$ .

$$\begin{aligned} 20 &= \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}} \left( 0.8 \frac{dx}{dt} + (0.6)(-60) \right) \\ \frac{dx}{dt} &= \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70 \end{aligned}$$

At the moment in question, the car's speed is 70 mph. ■



**FIGURE 3.47** The particle  $P$  travels clockwise along the circle (Example 4).

**EXAMPLE 4** A particle  $P$  moves clockwise at a constant rate along a circle of radius 10 m centered at the origin. The particle's initial position is  $(0, 10)$  on the  $y$ -axis, and its final destination is the point  $(10, 0)$  on the  $x$ -axis. Once the particle is in motion, the tangent line at  $P$  intersects the  $x$ -axis at a point  $Q$  (which moves over time). If it takes the particle 30 sec to travel from start to finish, how fast is the point  $Q$  moving along the  $x$ -axis when it is 20 m from the center of the circle?

**Solution** We picture the situation in the coordinate plane with the circle centered at the origin (see Figure 3.47). We let  $t$  represent time and let  $\theta$  denote the angle from the  $x$ -axis to the radial line joining the origin to  $P$ . Since the particle travels from start to finish in 30 sec, it is traveling along the circle at a constant rate of  $\pi/2$  radians in  $1/2$  min, or  $\pi$  rad/min. In other words,  $d\theta/dt = -\pi$ , with  $t$  being measured in minutes. The negative sign appears because  $\theta$  is decreasing over time.

Setting  $x(t)$  to be the distance at time  $t$  from the point  $Q$  to the origin, we want to find  $dx/dt$  when

$$x = 20 \text{ m} \quad \text{and} \quad \frac{d\theta}{dt} = -\pi \text{ rad/min.}$$

To relate the variables  $x$  and  $\theta$ , we see from Figure 3.47 that  $x \cos \theta = 10$ , or  $x = 10 \sec \theta$ . Differentiation of this last equation gives

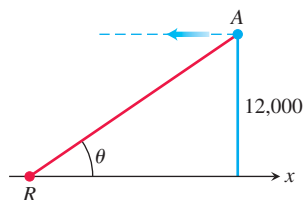
$$\frac{dx}{dt} = 10 \sec \theta \tan \theta \frac{d\theta}{dt} = -10\pi \sec \theta \tan \theta.$$

Note that  $dx/dt$  is negative because  $x$  is decreasing ( $Q$  is moving toward the origin).

When  $x = 20$ ,  $\cos \theta = 1/2$  and  $\sec \theta = 2$ . Also,  $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{3}$ . It follows that

$$\frac{dx}{dt} = (-10\pi)(2)(\sqrt{3}) = -20\sqrt{3}\pi.$$

At the moment in question, the point  $Q$  is moving toward the origin at the speed of  $20\sqrt{3}\pi \approx 109$  m/min. ■



**FIGURE 3.48** Jet airliner  $A$  traveling at constant altitude toward radar station  $R$  (Example 5).

**EXAMPLE 5** A jet airliner is flying at a constant altitude of 12,000 ft above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is  $30^\circ$ . How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of  $2/3$  deg/sec in order to keep the aircraft within its direct line of sight?

**Solution** The aircraft  $A$  and radar station  $R$  are pictured in the coordinate plane, using the positive  $x$ -axis as the horizontal distance at sea level from  $R$  to  $A$ , and the positive  $y$ -axis as the vertical altitude above sea level. We let  $t$  represent time and observe that  $y = 12,000$  is a constant. The general situation and line-of-sight angle  $\theta$  are depicted in Figure 3.48. We want to find  $dx/dt$  when  $\theta = \pi/6$  rad and  $d\theta/dt = 2/3$  deg/sec.

From Figure 3.48, we see that

$$\frac{12,000}{x} = \tan \theta \quad \text{or} \quad x = 12,000 \cot \theta.$$

Using miles instead of feet for our distance units, the last equation translates to

$$x = \frac{12,000}{5280} \cot \theta.$$

Differentiation with respect to  $t$  gives

$$\frac{dx}{dt} = -\frac{1200}{528} \csc^2 \theta \frac{d\theta}{dt}.$$

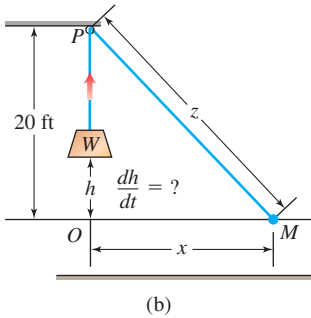
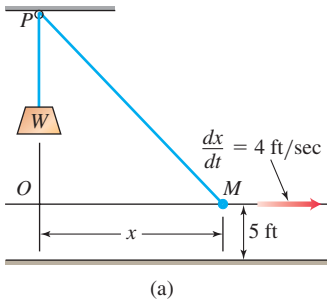
When  $\theta = \pi/6$ ,  $\sin^2 \theta = 1/4$ , so  $\csc^2 \theta = 4$ . Converting  $d\theta/dt = 2/3$  deg/sec to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{2}{3} \left( \frac{\pi}{180} \right) (3600) \text{ rad/hr.} \quad \text{1 hr} = 3600 \text{ sec, 1 deg} = \pi/180 \text{ rad}$$

Substitution into the equation for  $dx/dt$  then gives

$$\frac{dx}{dt} = \left( -\frac{1200}{528} \right) (4) \left( \frac{2}{3} \right) \left( \frac{\pi}{180} \right) (3600) \approx -380.$$

The negative sign appears because the distance  $x$  is decreasing, so the aircraft is approaching the island at a speed of approximately 380 mi/hr when first detected by the radar. ■



**FIGURE 3.49** A worker at  $M$  walks to the right, pulling the weight  $W$  upward as the rope moves through the pulley  $P$  (Example 6).

**EXAMPLE 6** Figure 3.49a shows a rope running through a pulley at  $P$  and bearing a weight  $W$  at one end. The other end is held 5 ft above the ground in the hand  $M$  of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking rapidly away from the vertical line  $PW$  at the rate of 4 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from  $PW$ ?

**Solution** We let  $OM$  be the horizontal line of length  $x$  ft from a point  $O$  directly below the pulley to the worker's hand  $M$  at any instant of time (Figure 3.49). Let  $h$  be the height of the weight  $W$  above  $O$ , and let  $z$  denote the length of rope from the pulley  $P$  to the worker's hand. We want to know  $dh/dt$  when  $x = 21$  given that  $dx/dt = 4$ . Note that the height of  $P$  above  $O$  is 20 ft because  $O$  is 5 ft above the ground. We assume the angle at  $O$  is a right angle.

At any instant of time  $t$  we have the following relationships (see Figure 3.49b):

$$20 - h + z = 45 \quad \text{Total length of rope is 45 ft.}$$

$$20^2 + x^2 = z^2. \quad \text{Angle at } O \text{ is a right angle.}$$

If we solve for  $z = 25 + h$  in the first equation, and substitute into the second equation, we have

$$20^2 + x^2 = (25 + h)^2. \quad (1)$$

Differentiating both sides with respect to  $t$  gives

$$2x \frac{dx}{dt} = 2(25 + h) \frac{dh}{dt},$$

and solving this last equation for  $dh/dt$  we find

$$\frac{dh}{dt} = \frac{x}{25 + h} \frac{dx}{dt}. \quad (2)$$

Since we know  $dx/dt$ , it remains only to find  $25 + h$  at the instant when  $x = 21$ . From Equation (1),

$$20^2 + 21^2 = (25 + h)^2$$

so that

$$(25 + h)^2 = 841, \quad \text{or} \quad 25 + h = 29.$$

Equation (2) now gives

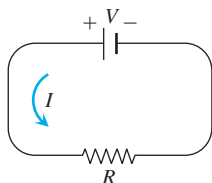
$$\frac{dh}{dt} = \frac{21}{29} \cdot 4 = \frac{84}{29} \approx 2.9 \text{ ft/sec}$$

as the rate at which the weight is being raised when  $x = 21$  ft. ■

## Exercises 3.10

- Area** Suppose that the radius  $r$  and area  $A = \pi r^2$  of a circle are differentiable functions of  $t$ . Write an equation that relates  $dA/dt$  to  $dr/dt$ .
- Surface area** Suppose that the radius  $r$  and surface area  $S = 4\pi r^2$  of a sphere are differentiable functions of  $t$ . Write an equation that relates  $dS/dt$  to  $dr/dt$ .
- Assume that  $y = 5x$  and  $dx/dt = 2$ . Find  $dy/dt$ .
- Assume that  $2x + 3y = 12$  and  $dy/dt = -2$ . Find  $dx/dt$ .
- If  $y = x^2$  and  $dx/dt = 3$ , then what is  $dy/dt$  when  $x = -1$ ?
- If  $x = y^3 - y$  and  $dy/dt = 5$ , then what is  $dx/dt$  when  $y = 2$ ?
- If  $x^2 + y^2 = 25$  and  $dx/dt = -2$ , then what is  $dy/dt$  when  $x = 3$  and  $y = -4$ ?
- If  $x^2 y^3 = 4/27$  and  $dy/dt = 1/2$ , then what is  $dx/dt$  when  $x = 2$ ?
- If  $L = \sqrt{x^2 + y^2}$ ,  $dx/dt = -1$ , and  $dy/dt = 3$ , find  $dL/dt$  when  $x = 5$  and  $y = 12$ .
- If  $r + s^2 + v^3 = 12$ ,  $dr/dt = 4$ , and  $ds/dt = -3$ , find  $dv/dt$  when  $r = 3$  and  $s = 1$ .

11. If the original 24 m edge length  $x$  of a cube decreases at the rate of 5 m/min, when  $x = 3$  m at what rate does the cube's
- surface area change?
  - volume change?
12. A cube's surface area increases at the rate of  $72 \text{ in}^2/\text{sec}$ . At what rate is the cube's volume changing when the edge length is  $x = 3$  in?
13. **Volume** The radius  $r$  and height  $h$  of a right circular cylinder are related to the cylinder's volume  $V$  by the formula  $V = \pi r^2 h$ .
- How is  $dV/dt$  related to  $dh/dt$  if  $r$  is constant?
  - How is  $dV/dt$  related to  $dr/dt$  if  $h$  is constant?
  - How is  $dV/dt$  related to  $dr/dt$  and  $dh/dt$  if neither  $r$  nor  $h$  is constant?
14. **Volume** The radius  $r$  and height  $h$  of a right circular cone are related to the cone's volume  $V$  by the equation  $V = (1/3)\pi r^2 h$ .
- How is  $dV/dt$  related to  $dh/dt$  if  $r$  is constant?
  - How is  $dV/dt$  related to  $dr/dt$  if  $h$  is constant?
  - How is  $dV/dt$  related to  $dr/dt$  and  $dh/dt$  if neither  $r$  nor  $h$  is constant?
15. **Changing voltage** The voltage  $V$  (volts), current  $I$  (amperes), and resistance  $R$  (ohms) of an electric circuit like the one shown here are related by the equation  $V = IR$ . Suppose that  $V$  is increasing at the rate of 1 volt/sec while  $I$  is decreasing at the rate of  $1/3$  amp/sec. Let  $t$  denote time in seconds.



- What is the value of  $dV/dt$ ?
  - What is the value of  $dI/dt$ ?
  - What equation relates  $dR/dt$  to  $dV/dt$  and  $dI/dt$ ?
  - Find the rate at which  $R$  is changing when  $V = 12$  volts and  $I = 2$  amps. Is  $R$  increasing, or decreasing?
16. **Electrical power** The power  $P$  (watts) of an electric circuit is related to the circuit's resistance  $R$  (ohms) and current  $I$  (amperes) by the equation  $P = RI^2$ .
- How are  $dP/dt$ ,  $dR/dt$ , and  $dI/dt$  related if none of  $P$ ,  $R$ , and  $I$  are constant?
  - How is  $dR/dt$  related to  $dI/dt$  if  $P$  is constant?
17. **Distance** Let  $x$  and  $y$  be differentiable functions of  $t$  and let  $s = \sqrt{x^2 + y^2}$  be the distance between the points  $(x, 0)$  and  $(0, y)$  in the  $xy$ -plane.
- How is  $ds/dt$  related to  $dx/dt$  if  $y$  is constant?
  - How is  $ds/dt$  related to  $dx/dt$  and  $dy/dt$  if neither  $x$  nor  $y$  is constant?
  - How is  $dx/dt$  related to  $dy/dt$  if  $s$  is constant?
18. **Diagonals** If  $x$ ,  $y$ , and  $z$  are lengths of the edges of a rectangular box, the common length of the box's diagonals is  $s = \sqrt{x^2 + y^2 + z^2}$ .

- Assuming that  $x$ ,  $y$ , and  $z$  are differentiable functions of  $t$ , how is  $ds/dt$  related to  $dx/dt$ ,  $dy/dt$ , and  $dz/dt$ ?
- How is  $ds/dt$  related to  $dy/dt$  and  $dz/dt$  if  $x$  is constant?
- How are  $dx/dt$ ,  $dy/dt$ , and  $dz/dt$  related if  $s$  is constant?

19. **Area** The area  $A$  of a triangle with sides of lengths  $a$  and  $b$  enclosing an angle of measure  $\theta$  is

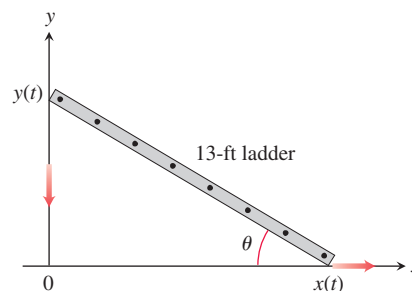
$$A = \frac{1}{2}ab \sin \theta.$$

- How is  $dA/dt$  related to  $d\theta/dt$  if  $a$  and  $b$  are constant?
  - How is  $dA/dt$  related to  $d\theta/dt$  and  $da/dt$  if only  $b$  is constant?
  - How is  $dA/dt$  related to  $d\theta/dt$ ,  $da/dt$ , and  $db/dt$  if none of  $a$ ,  $b$ , and  $\theta$  are constant?
20. **Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?
21. **Changing dimensions in a rectangle** The length  $l$  of a rectangle is decreasing at the rate of 2 cm/sec while the width  $w$  is increasing at the rate of 2 cm/sec. When  $l = 12$  cm and  $w = 5$  cm, find the rates of change of (a) the area, (b) the perimeter, and (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?
22. **Changing dimensions in a rectangular box** Suppose that the edge lengths  $x$ ,  $y$ , and  $z$  of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

Find the rates at which the box's (a) volume, (b) surface area, and (c) diagonal length  $s = \sqrt{x^2 + y^2 + z^2}$  are changing at the instant when  $x = 4$ ,  $y = 3$ , and  $z = 2$ .

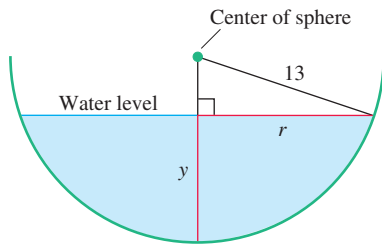
23. **A sliding ladder** A 13-ft ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.
- How fast is the top of the ladder sliding down the wall then?
  - At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
  - At what rate is the angle  $\theta$  between the ladder and the ground changing then?



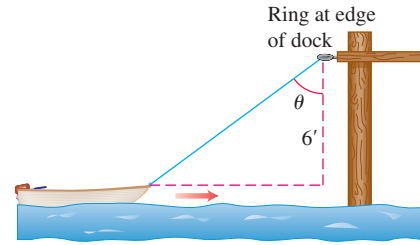
24. **Commercial air traffic** Two commercial airplanes are flying at an altitude of 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5

nautical miles from the intersection point and  $B$  is 12 nautical miles from the intersection point?

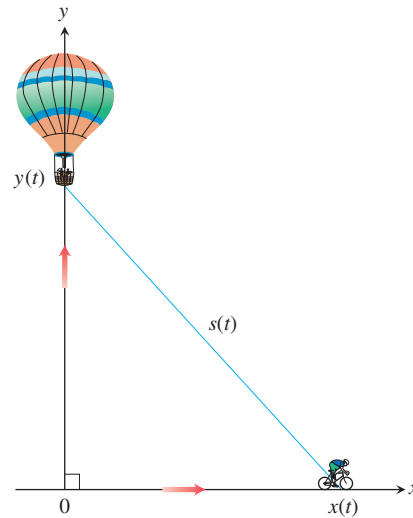
25. **Flying a kite** A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?
26. **Boring a cylinder** The mechanics at Lincoln Automotive are reboring a 6-in.-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?
27. **A growing sand pile** Sand falls from a conveyor belt at the rate of  $10 \text{ m}^3/\text{min}$  onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the (a) height and (b) radius changing when the pile is 4 m high? Answer in centimeters per minute.
28. **A draining conical reservoir** Water is flowing at the rate of  $50 \text{ m}^3/\text{min}$  from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.
- How fast (centimeters per minute) is the water level falling when the water is 5 m deep?
  - How fast is the radius of the water's surface changing then? Answer in centimeters per minute.
29. **A draining hemispherical reservoir** Water is flowing at the rate of  $6 \text{ m}^3/\text{min}$  from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius  $R$  is  $V = (\pi/3)y^2(3R - y)$  when the water is  $y$  meters deep.



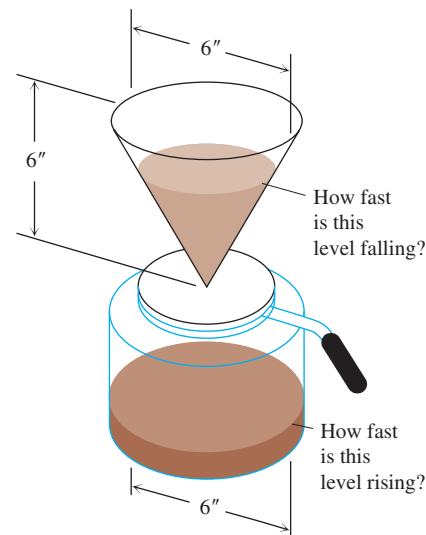
- At what rate is the water level changing when the water is 8 m deep?
  - What is the radius  $r$  of the water's surface when the water is  $y$  m deep?
  - At what rate is the radius  $r$  changing when the water is 8 m deep?
30. **A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.
31. **The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of  $100\pi \text{ ft}^3/\text{min}$ . How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?
32. **Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec.
- How fast is the boat approaching the dock when 10 ft of rope are out?
  - At what rate is the angle  $\theta$  changing at this instant (see the figure)?



33. **A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance  $s(t)$  between the bicycle and balloon increasing 3 sec later?



34. **Making coffee** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of  $10 \text{ in}^3/\text{min}$ .
- How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?
  - How fast is the level in the cone falling then?





- 35. Cardiac output** In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzburg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

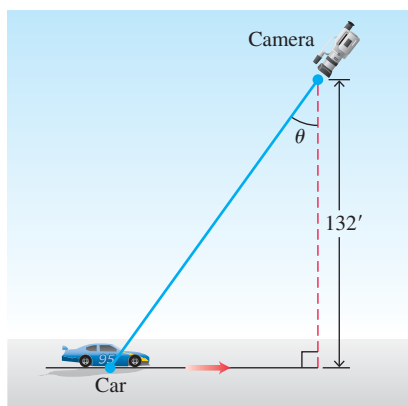
where  $Q$  is the number of milliliters of  $\text{CO}_2$  you exhale in a minute and  $D$  is the difference between the  $\text{CO}_2$  concentration (ml/L) in the blood pumped to the lungs and the  $\text{CO}_2$  concentration in the blood returning from the lungs. With  $Q = 233$  ml/min and  $D = 97 - 56 = 41$  ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

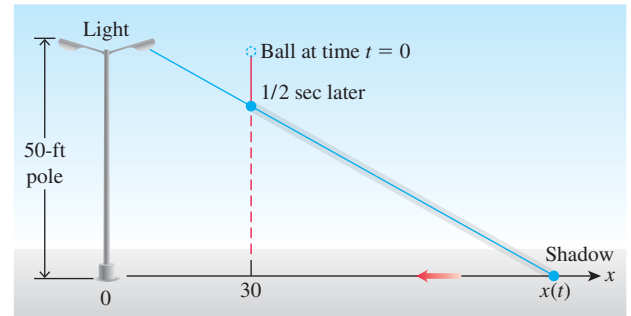
fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when  $Q = 233$  and  $D = 41$ , we also know that  $D$  is decreasing at the rate of 2 units a minute but that  $Q$  remains unchanged. What is happening to the cardiac output?

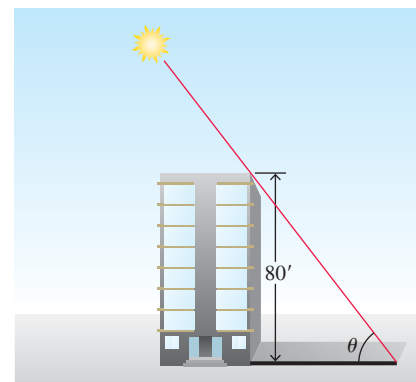
- 36. Moving along a parabola** A particle moves along the parabola  $y = x^2$  in the first quadrant in such a way that its  $x$ -coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination  $\theta$  of the line joining the particle to the origin changing when  $x = 3$  m?
- 37. Motion in the plane** The coordinates of a particle in the metric  $xy$ -plane are differentiable functions of time  $t$  with  $dx/dt = -1$  m/sec and  $dy/dt = -5$  m/sec. How fast is the particle's distance from the origin changing as it passes through the point  $(5, 12)$ ?
- 38. Videotaping a moving car** You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mi/h (264 ft/sec), as shown in the accompanying figure. How fast will your camera angle  $\theta$  be changing when the car is right in front of you? A half second later?



- 39. A moving shadow** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance  $s = 16t^2$  ft in  $t$  sec.)

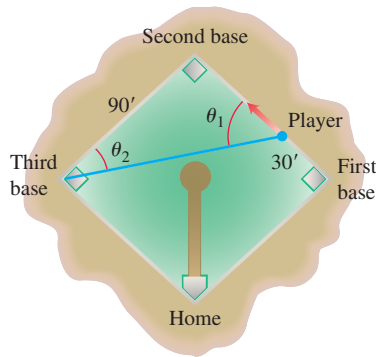


- 40. A building's shadow** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle  $\theta$  the sun makes with the ground is increasing at the rate of  $0.27^\circ/\text{min}$ . At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)



- 41. A melting ice layer** A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of  $10 \text{ in}^3/\text{min}$ , how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?
- 42. Highway patrol** A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi, the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.
- 43. Baseball players** A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec.
- At what rate is the player's distance from third base changing when the player is 30 ft from first base?
  - At what rates are angles  $\theta_1$  and  $\theta_2$  (see the figure) changing at that time?

- c. The player slides into second base at the rate of 15 ft/sec. At what rates are angles  $\theta_1$  and  $\theta_2$  changing as the player touches base?



44. **Ships** Two ships are steaming straight away from a point  $O$  along routes that make a  $120^\circ$  angle. Ship  $A$  moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship  $B$  moves at 21 knots. How fast are the ships moving apart when  $OA = 5$  and  $OB = 3$  nautical miles?
45. **Clock's moving hands** At what rate is the angle between a clock's minute and hour hands changing at 4 o'clock in the afternoon?
46. **Oil spill** An explosion at an oil rig located in gulf waters causes an elliptical oil slick to spread on the surface from the rig. The slick is a constant 9 in. thick. After several days, when the major axis of the slick is 2 mi long and the minor axis is  $3/4$  mi wide, it is determined that its length is increasing at the rate of 30 ft/hr, and its width is increasing at the rate of 10 ft/hr. At what rate (in cubic feet per hour) is oil flowing from the site of the rig at that time?

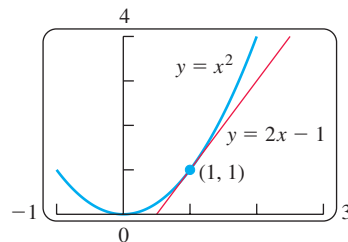
## 3.11 Linearization and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called *linearizations*, and they are based on tangent lines. Other approximating functions, such as polynomials, are discussed in Chapter 10.

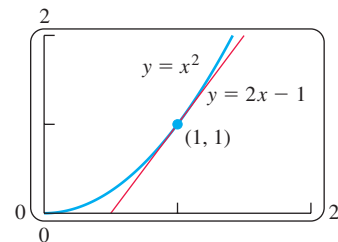
We introduce new variables  $dx$  and  $dy$ , called *differentials*, and define them in a way that makes Leibniz's notation for the derivative  $dy/dx$  a true ratio. We use  $dy$  to estimate error in measurement, which then provides for a precise proof of the Chain Rule (Section 3.6).

### Linearization

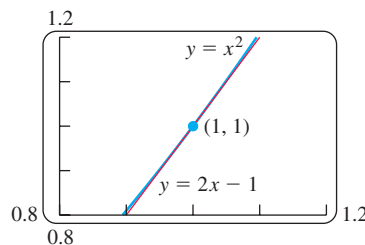
As you can see in Figure 3.50, the tangent to the curve  $y = x^2$  lies close to the curve near the point of tangency. For a brief interval to either side, the  $y$ -values along the tangent line



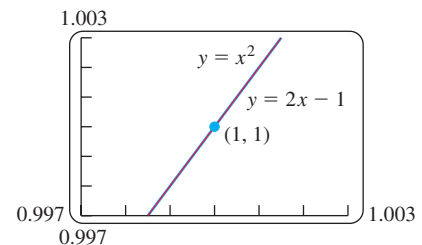
$y = x^2$  and its tangent  $y = 2x - 1$  at  $(1, 1)$ .



Tangent and curve very close near  $(1, 1)$ .



Tangent and curve very close throughout entire  $x$ -interval shown.



Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this  $x$ -interval.

**FIGURE 3.50** The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.