

4

Applications of Derivatives

OVERVIEW One of the most important applications of the derivative is its use as a tool for finding the optimal (best) solutions to problems. Optimization problems abound in mathematics, physical science and engineering, business and economics, and biology and medicine. For example, what are the height and diameter of the cylinder of largest volume that can be inscribed in a given sphere? What are the dimensions of the strongest rectangular wooden beam that can be cut from a cylindrical log of given diameter? Based on production costs and sales revenue, how many items should a manufacturer produce to maximize profit? How much does the trachea (windpipe) contract to expel air at the maximum speed during a cough? What is the branching angle at which blood vessels minimize the energy loss due to friction as blood flows through the branches?

In this chapter we use derivatives to find extreme values of functions, to determine and analyze the shapes of graphs, and to solve equations numerically. We also introduce the idea of recovering a function from its derivative. The key to many of these applications is the Mean Value Theorem, which paves the way to integral calculus.

4.1 Extreme Values of Functions

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Once we can do this, we can solve a variety of optimization problems (see Section 4.6). The domains of the functions we consider are intervals or unions of separate intervals.

DEFINITIONS Let f be a function with domain D . Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \quad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } D.$$

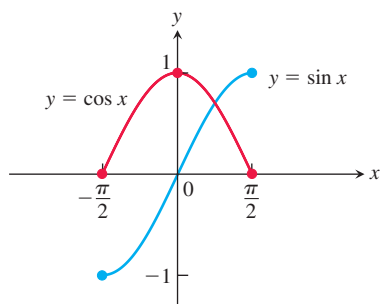


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Maximum and minimum values are called **extreme values** of the function f . Absolute maxima or minima are also referred to as **global** maxima or minima.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1).

Functions with the same defining rule or formula can have different extrema (maximum or minimum values), depending on the domain. We see this in the following example.

EXAMPLE 1 The absolute extrema of the following functions on their domains can be seen in Figure 4.2. Each function has the same defining equation, $y = x^2$, but the domains vary. Notice that a function might not have a maximum or minimum if the domain is unbounded or fails to contain an endpoint.

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema

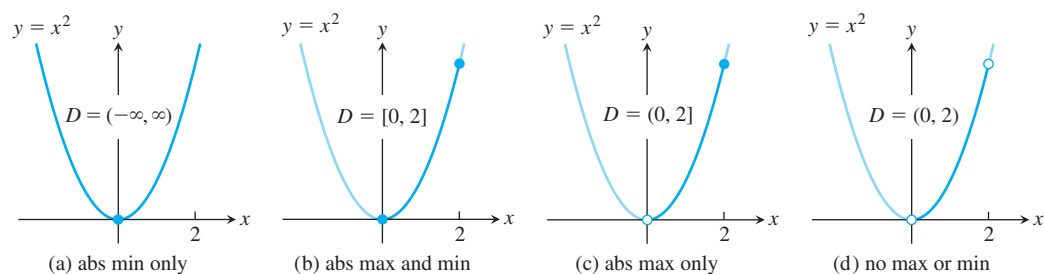


FIGURE 4.2 Graphs for Example 1.

HISTORICAL BIOGRAPHY

Daniel Bernoulli
(1700–1789)

Some of the functions in Example 1 did not have a maximum or a minimum value. The following theorem asserts that a function which is *continuous* over (or on) a finite *closed* interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We look for these extreme values when we graph a function.

THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

The proof of the Extreme Value Theorem requires a detailed knowledge of the real number system (see Appendix 7) and we will not give it here. Figure 4.3 illustrates possible locations for the absolute extrema of a continuous function on a closed interval $[a, b]$. As we observed for the function $y = \cos x$, it is possible that an absolute minimum (or absolute maximum) may occur at two or more different points of the interval.

The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are key ingredients. Without them, the conclusion of the theorem

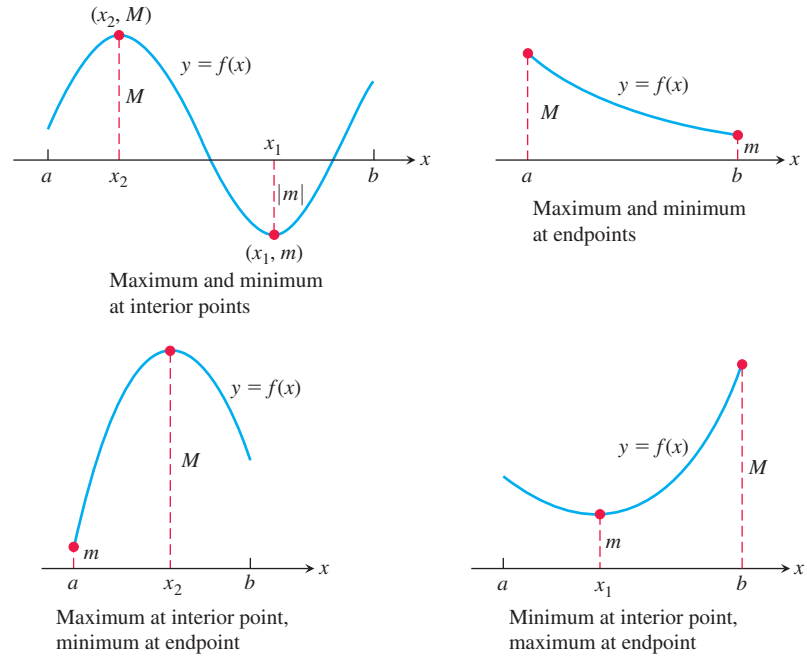


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

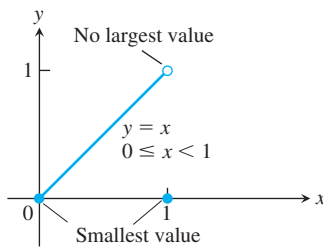


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

need not hold. Example 1 shows that an absolute extreme value may not exist if the interval fails to be both closed and finite. The exponential function $y = e^x$ over $(-\infty, \infty)$ shows that neither extreme value need exist on an infinite interval. Figure 4.4 shows that the continuity requirement cannot be omitted.

Local (Relative) Extreme Values

Figure 4.5 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby*. The curve rises to the left and falls to the right around c , making $f(c)$ a maximum locally. The function attains its absolute maximum at d . We now define what we mean by local extrema.

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c .

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c .

If the domain of f is the closed interval $[a, b]$, then f has a local maximum at the endpoint $x = a$, if $f(x) \leq f(a)$ for all x in some half-open interval $[a, a + \delta)$, $\delta > 0$. Likewise, f has a local maximum at an interior point $x = c$ if $f(x) \leq f(c)$ for all x in some open interval $(c - \delta, c + \delta)$, $\delta > 0$, and a local maximum at the endpoint $x = b$ if $f(x) \leq f(b)$ for all x in some half-open interval $(b - \delta, b]$, $\delta > 0$. The inequalities are reversed for local minimum values. In Figure 4.5, the function f has local maxima at c and d and local minima at a , e , and b . Local extrema are also called **relative extrema**. Some functions can have infinitely many local extrema, even over a finite interval. One example is the function $f(x) = \sin(1/x)$ on the interval $(0, 1]$. (We graphed this function in Figure 2.40.)

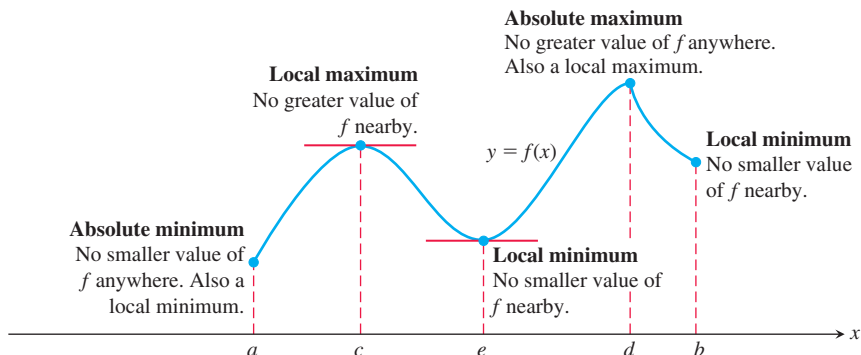


FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one.* Similarly, *a list of all local minima will include the absolute minimum if there is one.*

Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function’s extrema.

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c , then

$$f'(c) = 0.$$

Proof To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Figure 4.6) so that $f(x) - f(c) \leq 0$ for all values of x near enough to c . Since c is an interior point of f ’s domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \leq 0. \quad \text{Because } (x - c) > 0 \text{ and } f(x) \leq f(c) \quad (1)$$

Similarly,

$$f'(c) = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \quad \text{Because } (x - c) < 0 \text{ and } f(x) \leq f(c) \quad (2)$$

Together, Equations (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$, which reverses the inequalities in Equations (1) and (2). ■

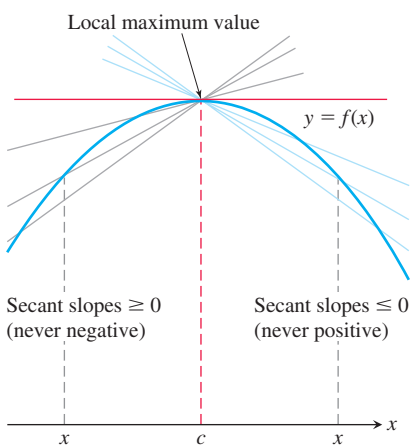


FIGURE 4.6 A curve with a local maximum value. The slope at c , simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

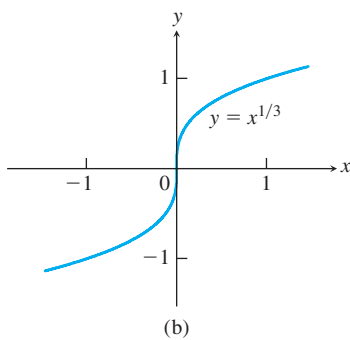
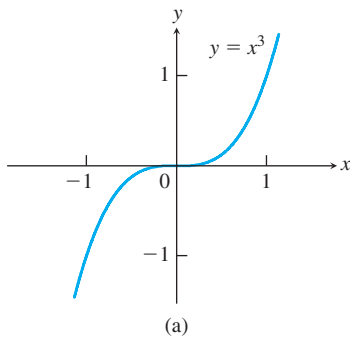


FIGURE 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. If we recall that all the domains we consider are intervals or unions of separate intervals, the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$, At $x = c$ and $x = e$ in Fig. 4.5
2. interior points where f' is undefined, At $x = d$ in Fig. 4.5
3. endpoints of the domain of f . At $x = a$ and $x = b$ in Fig. 4.5

The following definition helps us to summarize these results.

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f .

Thus the only domain points where a function can assume extreme values are critical points and endpoints. However, be careful not to misinterpret what is being said here. A function may have a critical point at $x = c$ without having a local extreme value there. For instance, both of the functions $y = x^3$ and $y = x^{1/3}$ have critical points at the origin, but neither function has a local extreme value at the origin. Instead, each function has a *point of inflection* there (see Figure 4.7). We define and explore inflection points in Section 4.4.

Most problems that ask for extreme values call for finding the absolute extrema of a continuous function on a closed and finite interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. Often we can simply list these points and calculate the corresponding function values to find what the largest and smallest values are, and where they are located. Of course, if the interval is not closed or not finite (such as $a < x < b$ or $a < x < \infty$), we have seen that absolute extrema need not exist. If an absolute maximum or minimum value does exist, it must occur at a critical point or at an included right- or left-hand endpoint of the interval.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXAMPLE 2 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

$$\text{Critical point value: } f(0) = 0$$

$$\text{Endpoint values: } f(-2) = 4$$

$$f(1) = 1.$$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

EXAMPLE 3 Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

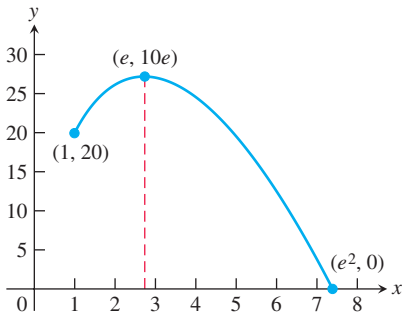


FIGURE 4.8 The extreme values of $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$ occur at $x = e$ and $x = e^2$ (Example 3).

Solution Figure 4.8 suggests that f has its absolute maximum value near $x = 3$ and its absolute minimum value of 0 at $x = e^2$. Let's verify this observation.

We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative is

$$f'(x) = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 10(1 - \ln x).$$

The only critical point in the domain $[1, e^2]$ is the point $x = e$, where $\ln x = 1$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(e) = 10e$

Endpoint values: $f(1) = 10(2 - \ln 1) = 20$

$f(e^2) = 10e^2(2 - 2 \ln e) = 0.$

We can see from this list that the function's absolute maximum value is $10e \approx 27.2$; it occurs at the critical interior point $x = e$. The absolute minimum value is 0 and occurs at the right endpoint $x = e^2$. ■

EXAMPLE 4 Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{9}.$

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp (Figure 4.9). ■

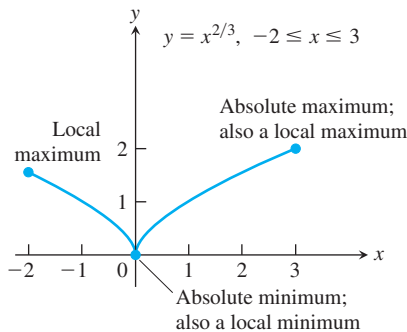
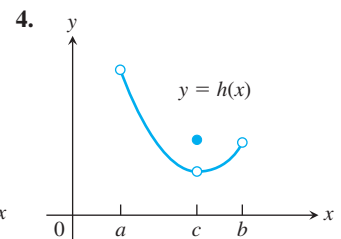
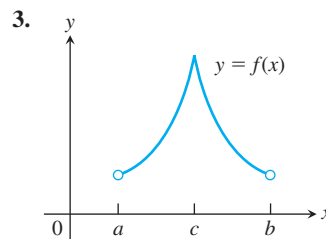
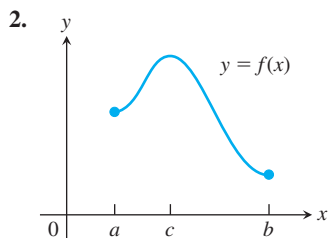
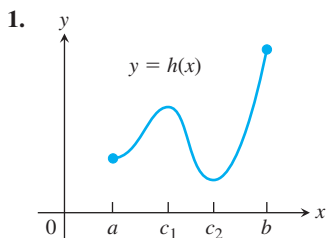


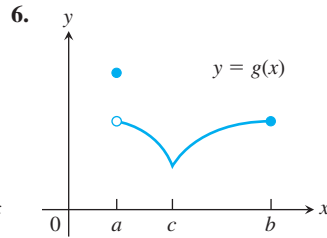
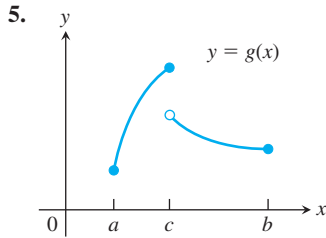
FIGURE 4.9 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

Exercises 4.1

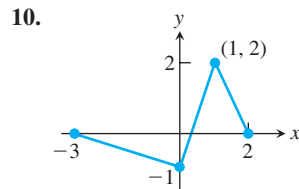
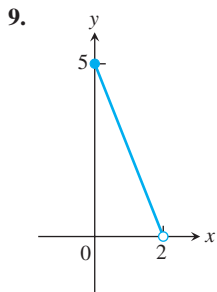
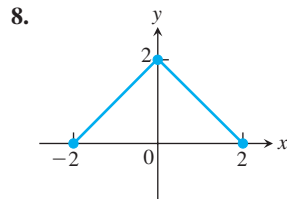
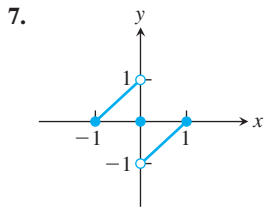
Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1





In Exercises 7–10, find the absolute extreme values and where they occur.



In Exercises 11–14, match the table with a graph.

11.

x	$f'(x)$
a	0
b	0
c	5

12.

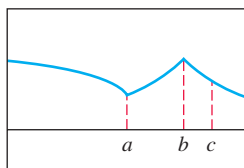
x	$f'(x)$
a	0
b	0
c	-5

13.

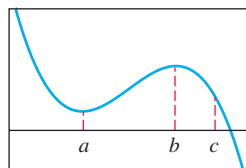
x	$f'(x)$
a	does not exist
b	0
c	-2

14.

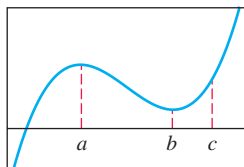
x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



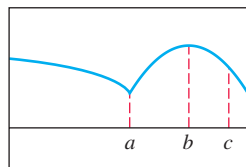
(a)



(b)



(c)



(d)

In Exercises 15–20, sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with Theorem 1.

15. $f(x) = |x|, -1 < x < 2$

16. $y = \frac{6}{x^2 + 2}, -1 < x < 1$

17. $g(x) = \begin{cases} -x, & 0 \leq x < 1 \\ x - 1, & 1 \leq x \leq 2 \end{cases}$

18. $h(x) = \begin{cases} \frac{1}{x}, & -1 \leq x < 0 \\ \sqrt{x}, & 0 \leq x \leq 4 \end{cases}$

19. $y = 3 \sin x, 0 < x < 2\pi$

20. $f(x) = \begin{cases} x + 1, & -1 \leq x < 0 \\ \cos x, & 0 < x \leq \frac{\pi}{2} \end{cases}$

Absolute Extrema on Finite Closed Intervals

In Exercises 21–40, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

21. $f(x) = \frac{2}{3}x - 5, -2 \leq x \leq 3$

22. $f(x) = -x - 4, -4 \leq x \leq 1$

23. $f(x) = x^2 - 1, -1 \leq x \leq 2$

24. $f(x) = 4 - x^3, -2 \leq x \leq 1$

25. $F(x) = -\frac{1}{x^2}, 0.5 \leq x \leq 2$

26. $F(x) = -\frac{1}{x}, -2 \leq x \leq -1$

27. $h(x) = \sqrt[3]{x}, -1 \leq x \leq 8$

28. $h(x) = -3x^{2/3}, -1 \leq x \leq 1$

29. $g(x) = \sqrt{4 - x^2}, -2 \leq x \leq 1$

30. $g(x) = -\sqrt{5 - x^2}, -\sqrt{5} \leq x \leq 0$

31. $f(\theta) = \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$

32. $f(\theta) = \tan \theta, -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{4}$

33. $g(x) = \csc x, \frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$

34. $g(x) = \sec x, -\frac{\pi}{3} \leq x \leq \frac{\pi}{6}$

35. $f(t) = 2 - |t|, -1 \leq t \leq 3$

36. $f(t) = |t - 5|, 4 \leq t \leq 7$

37. $g(x) = xe^{-x}, -1 \leq x \leq 1$

38. $h(x) = \ln(x + 1), 0 \leq x \leq 3$

39. $f(x) = \frac{1}{x} + \ln x, 0.5 \leq x \leq 4$

40. $g(x) = e^{-x^2}, -2 \leq x \leq 1$

In Exercises 41–44, find the function's absolute maximum and minimum values and say where they are assumed.

41. $f(x) = x^{4/3}$, $-1 \leq x \leq 8$
 42. $f(x) = x^{5/3}$, $-1 \leq x \leq 8$
 43. $g(\theta) = \theta^{3/5}$, $-32 \leq \theta \leq 1$
 44. $h(\theta) = 3\theta^{2/3}$, $-27 \leq \theta \leq 8$

Finding Critical Points

In Exercises 45–52, determine all critical points for each function.

45. $y = x^2 - 6x + 7$ 46. $f(x) = 6x^2 - x^3$
 47. $f(x) = x(4 - x)^3$ 48. $g(x) = (x - 1)^2(x - 3)^2$
 49. $y = x^2 + \frac{2}{x}$ 50. $f(x) = \frac{x^2}{x - 2}$
 51. $y = x^2 - 32\sqrt{x}$ 52. $g(x) = \sqrt{2x - x^2}$

Finding Extreme Values

In Exercises 53–68, find the extreme values (absolute and local) of the function over its natural domain, and where they occur.

53. $y = 2x^2 - 8x + 9$ 54. $y = x^3 - 2x + 4$
 55. $y = x^3 + x^2 - 8x + 5$ 56. $y = x^3(x - 5)^2$
 57. $y = \sqrt{x^2 - 1}$ 58. $y = x - 4\sqrt{x}$
 59. $y = \frac{1}{\sqrt[3]{1 - x^2}}$ 60. $y = \sqrt{3 + 2x - x^2}$
 61. $y = \frac{x}{x^2 + 1}$ 62. $y = \frac{x + 1}{x^2 + 2x + 3}$
 63. $y = e^x + e^{-x}$ 64. $y = e^x - e^{-x}$
 65. $y = x \ln x$ 66. $y = x^2 \ln x$
 67. $y = \cos^{-1}(x^2)$ 68. $y = \sin^{-1}(e^x)$

Local Extrema and Critical Points

In Exercises 69–76, find the critical points, domain endpoints, and extreme values (absolute and local) for each function.

69. $y = x^{2/3}(x + 2)$ 70. $y = x^{2/3}(x^2 - 4)$
 71. $y = x\sqrt{4 - x^2}$ 72. $y = x^2\sqrt{3 - x}$
 73. $y = \begin{cases} 4 - 2x, & x \leq 1 \\ x + 1, & x > 1 \end{cases}$ 74. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \geq 0 \end{cases}$
 75. $y = \begin{cases} -x^2 - 2x + 4, & x \leq 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$
 76. $y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \leq 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

In Exercises 77 and 78, give reasons for your answers.

77. Let $f(x) = (x - 2)^{2/3}$.
 a. Does $f'(2)$ exist?
 b. Show that the only local extreme value of f occurs at $x = 2$.
 c. Does the result in part (b) contradict the Extreme Value Theorem?
 d. Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a .
 78. Let $f(x) = |x^3 - 9x|$.
 a. Does $f'(0)$ exist? b. Does $f'(3)$ exist?
 c. Does $f'(-3)$ exist? d. Determine all extrema of f .

Theory and Examples

79. **A minimum with no derivative** The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though f is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.
 80. **Even functions** If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.
 81. **Odd functions** If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of g at $x = -c$? Give reasons for your answer.
 82. **No critical points or endpoints exist** We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

83. The function

$$V(x) = x(10 - 2x)(16 - 2x), \quad 0 < x < 5,$$

models the volume of a box.

- a. Find the extreme values of V .
 b. Interpret any values found in part (a) in terms of the volume of the box.
 84. **Cubic functions** Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$
 a. Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.
 b. How many local extreme values can f have?
 85. **Maximum height of a vertically moving body** The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \quad g > 0,$$

with s in meters and t in seconds. Find the body's maximum height.

86. **Peak alternating current** Suppose that at any given time t (in seconds) the current i (in amperes) in an alternating current circuit is $i = 2 \cos t + 2 \sin t$. What is the peak current for this circuit (largest magnitude)?

T Graph the functions in Exercises 87–90. Then find the extreme values of the function on the interval and say where they occur.

87. $f(x) = |x - 2| + |x + 3|$, $-5 \leq x \leq 5$
 88. $g(x) = |x - 1| - |x - 5|$, $-2 \leq x \leq 7$
 89. $h(x) = |x + 2| - |x - 3|$, $-\infty < x < \infty$
 90. $k(x) = |x + 1| + |x - 3|$, $-\infty < x < \infty$

COMPUTER EXPLORATIONS

In Exercises 91–98, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps.

- a. Plot the function over the interval to see its general behavior there.
 b. Find the interior points where $f' = 0$. (In some exercises, you may have to use the numerical equation solver to approximate a solution.) You may want to plot f' as well.
 c. Find the interior points where f' does not exist.

- d. Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.
- e. Find the function's absolute extreme values on the interval and identify where they occur.
91. $f(x) = x^4 - 8x^2 + 4x + 2$, $[-20/25, 64/25]$
92. $f(x) = -x^4 + 4x^3 - 4x + 1$, $[-3/4, 3]$
93. $f(x) = x^{2/3}(3 - x)$, $[-2, 2]$

94. $f(x) = 2 + 2x - 3x^{2/3}$, $[-1, 10/3]$
95. $f(x) = \sqrt{x} + \cos x$, $[0, 2\pi]$
96. $f(x) = x^{3/4} - \sin x + \frac{1}{2}$, $[0, 2\pi]$
97. $f(x) = \pi x^2 e^{-3x/2}$, $[0, 5]$
98. $f(x) = \ln(2x + x \sin x)$, $[1, 15]$

4.2 The Mean Value Theorem

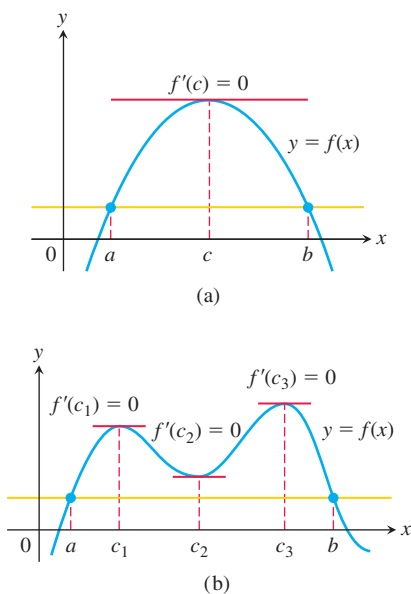


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

HISTORICAL BIOGRAPHY

Michel Rolle
(1652–1719)

We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these and other questions in this chapter by applying the Mean Value Theorem. First we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

Rolle's Theorem

As suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero (Figure 4.10). We now state and prove this result.

THEOREM 3—Rolle's Theorem Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b) . If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$ by Theorem 1. These can occur only

1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at endpoints of the function's domain, in this case a and b .

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b .

If either the maximum or the minimum occurs at a point c between a and b , then $f'(c) = 0$ by Theorem 2 in Section 4.1, and we have found a point for Rolle's Theorem.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b) . ■

The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Figure 4.11).

Rolle's Theorem may be combined with the Intermediate Value Theorem to show when there is only one real solution of an equation $f(x) = 0$, as we illustrate in the next example.

EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.