chapter "3"

Second order linear Differential equation

3.1 Homogeneous equation with constant coefficients. + 3.4 + 3.5

Linear 2^{nd} order D.E :-

$$\boxed{P(t)\,\ddot{y} + Q(t)\,\dot{y} + R(t)\,y = G(t)}$$

$\div P(t) \Rightarrow$

$$\boxed{\ddot{y} + p(t)\,\dot{y} + q(t)\,y = g(t)}$$

If $g(t) = 0 \rightarrow$ the equation is called homogeneous.

If $g(t) \neq 0 \rightarrow$ the equation is called nonhomogeneous.

⊛ I.V.P :

DE : $\ddot{y} + p(t)\,\dot{y} + q(t)\,y = g(t)$

with initial conditions $y(t_0) = y_0$

$\dot{y}(t_0) = \dot{y}_0$

linear homogeneous, second order D.E with constant coefficients.

$$\boxed{a y'' + b y' + c y = 0}, \quad a,b,c \in \mathbb{R}$$

$\hookrightarrow$ to solve this equation we use characteristic equation.

$y'' \rightarrow r^2$

$y' \rightarrow r^1$

$y \rightarrow r^0 = 1$

$$\boxed{a r^2 + b r + c = 0}$$

$\hookrightarrow$ we want to find (r)

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we have 3 cases:

1) real and different (distinct) root. $(b^2 - 4ac > 0)$

$r_1 \neq r_2$ (reals).

the sol:- $\quad y_1 = e^{r_1 t}$

$\quad y_2 = e^{r_2 t}$

the general solution: $y = c_1 y_1 + c_2 y_2$

2) one repeated root $(b^2 - 4ac = 0)$

$r_1 = r_2 = r$ (real)

the sol: $\quad y_1 = e^{rt}$

$\quad y_2 = t e^{rt}$

the general solution: $y = c_1 y_1 + c_2 y_2$

③ r_1, r_2 complex conjugate.

$(b^2 - 4ac < 0)$

$r_1 = \bar{r}_2 \rightarrow r_1 = \lambda + \mu i$

$r_2 = \lambda - \mu i$

the solutions : $y_1 = e^{\lambda t} \cos(\mu t)$

$y_2 = e^{\lambda t} \sin(\mu t)$

the general solution : (G.S)

$y = c_1 y_1 + c_2 y_2$.

example 1 (case 1)

solve the I.V.P $y'' + 5y' + 6y = 0$

$y(0) = 2$

$y'(0) = 3$.

Sol: $y'' + 5y' + 6y = 0$

char. eq. : $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, -3 \Rightarrow r_1 = -2, r_2 = -3$.

$y_1 = e^{-2t}, \quad y_2 = e^{-3t} \Rightarrow$

G.S : $y = c_1 e^{-2t} + c_2 e^{-3t}$

because we have I.C find c_1, c_2 :

$y(0) = 2 \Rightarrow$

$y = c_1 e^{-2t} + c_2 e^{-3t}$

$2 = c_1 e^{0} + c_2 e^{0} \rightarrow \boxed{2 = c_1 + c_2}$

$y'(0) = 3 \Rightarrow$

$y' = -2c_1 e^{-2t} + -3c_2 e^{-3t}$

$3 = -2c_1 e^{0} + -3c_2 e^{0} \rightarrow \boxed{3 = -2c_1 - 3c_2}$

$$\left.\begin{array}{l} c_1 + c_2 = 2 \\ -2c_1 - 3c_2 = 3 \end{array}\right\} \rightarrow \begin{array}{l} \cancel{2c_1} + 2c_2 = 4 \\ \cancel{-2c_1} - 3c_2 = 3 \\ \hline -c_2 = 7 \end{array}$$

$\rightarrow \boxed{c_2 = -7}$

$c_1 + c_2 = 2$

$c_1 - 7 = 2 \rightarrow \boxed{c_1 = 9}$

The solution is :-

$y = 9e^{-2t} - 7e^{-3t}$.

ex $4y'' - 8y' + 3y = 0$ solve. D.E

sol: $\rightarrow 4r^2 - 8r + 3 = 0$

$\rightarrow r = \dfrac{8 \pm \sqrt{64 - 4(4)(3)}}{2(4)}$

$= \dfrac{8 \pm \sqrt{16}}{8} = \dfrac{8 \pm 4}{8}$

$\rightarrow \dfrac{8+4}{8} = \dfrac{12}{8} = \dfrac{3}{2}$

$\rightarrow \dfrac{8-4}{8} = \dfrac{4}{8} = \dfrac{1}{2}$

$r_1 = \frac{3}{2}, \quad r_2 = \frac{1}{2}.$

$\rightarrow y_1 = e^{\frac{3}{2}t}, \quad y_2 = e^{\frac{1}{2}t} \rightarrow$ **G.S** $y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}$

example 2 (case 2):

find the general solution at the D.E

$y'' + 4y' + 4y = 0.$

sol: char. eq: $\rightarrow r^2 + 4r + 4 = 0$

$\rightarrow (r+2)(r+2) = 0 \rightarrow r = -2, -2$

$\rightarrow$ sol: $\rightarrow y_1 = e^{-2t}$

$y_2 = t e^{-2t}$

G.S: $y = c_1 e^{-2t} + c_2 t e^{-2t}.$

example (3) → (Case 3)

solve the DE $\ddot{y} + \dot{y} + y = 0$

sol: char. eq: $r^2 + r + 1 = 0$

$$r = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2}$$

$\sqrt{-1} = i$

complex number $\Rightarrow z = \lambda + \mu i$, $a, b \in R$

real part

imagenary part

conjugate :-

$z = a + bi \longrightarrow \bar{z} = a - bi$

ex $z = 1 + 3i \longrightarrow \bar{z} = 1 - 3i$

$z = 5 - 7i \longrightarrow \bar{z} = 5 + 7i$

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$$r = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{-1}\sqrt{3}}{2}$$

$$r = \frac{-1 \pm \sqrt{3}\, i}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

λ μ

$r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i \longrightarrow r_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$

$\lambda = \frac{-1}{2}$, $\mu = \frac{\sqrt{3}}{2}$ $\quad r_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2} i$

$\Rightarrow y_1 = e^{\frac{-1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right)$

$y_2 = e^{\frac{-1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$

G.S $y = c_1 e^{\frac{-1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{\frac{-1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$

ex $\ddot{y} + 9y = 0$ solve the D.E

sol: $\longrightarrow r^2 + 9 = 0$

$\longrightarrow r^2 = -9 \longrightarrow r = \pm\sqrt{-9}$

$r = \pm\sqrt{9}\sqrt{-1} \longrightarrow r = \pm 3i$

$\longrightarrow r = 0 \pm 3i$, $\lambda = 0$, $\mu = 3$

sol: $y_1 = e^{0t} \cos(3t) = \cos(3t)$

$y_2 = e^{0t} \sin(3t) = \sin(3t)$.

G.S: $y = c_1 \cos 3t + c_2 \sin 3t$

ex solve the I.V.P

$y'' - y = 0$, $y(0) = 2$

$y'(0) = -1$

$y = \frac{1}{2} e^{t} + \frac{3}{2} e^{-t}$

ex $y'' + y = 0$

$\rightarrow r^2 + 1 = 0 \rightarrow r^2 = -1$

$\rightarrow r = \pm\sqrt{-1} \rightarrow r = \pm i = 0 \pm i$ λ $\mu = 1$

$\rightarrow$ G.S $y_1 = e^{0t}\cos(t) = \cos t$

$y_2 = e^{0t}\sin(t) = \sin t$

$y = c_1 \cos t + c_2 \sin t.$