3.2 Fundamental Solutions of linear homogeneous equations

Thm

consider the I.V.P

$$y'' + p(t)\,y' + q(t)\,y = g(t), \quad y(t_0) = y_0$$
$$y'(t_0) = y'_0$$

where p, q, g are continuous on an open interval (I), then there exist exactly one (unique) solution $y = \phi(t)$ and the solution exists throughout the interval I.

ex Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)\,y'' + t\,y' - (t+3)\,y = 0, \quad y(1) = 2, \; y'(1) = 1$$

is certain to exist.

(unique solution. صيغة أخرى للسؤال)

Sol: $\div (t^2 - 3t)$ $\qquad 1 = y''$ معامل

$$y'' + \frac{t}{t^2 - 3t}\,y' - \frac{(t+3)}{t^2 - 3t}\,y = 0$$

$$p(t) = \frac{t}{t^2 - 3t} = \frac{t}{t(t-3)} = \frac{1}{t-3} \,.$$

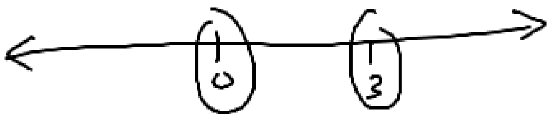
contin on $R - \{3\}$

$q(t) = \dfrac{-(t+3)}{t^2-3t} = \dfrac{-(t+3)}{t(t-3)} \Rightarrow$ cont. on $R - \{0,3\}$

$g(t) = 0 \rightarrow$ continuous on R

p, q, g are continuous on :-

$R - \{0,3\} \longrightarrow (-\infty, 0) \cup (0,3) \cup (3, \infty)$



initial condition $y(1) = 2$, t_0

$y'(1) = 1$

there exist a unique solution on $(0,3)$

continuous fun:

Poly $\longrightarrow f(t) = t^2 + 1$ ✓ cont. on R

Rational $\dfrac{P_1(t)}{P_2(t)} \longrightarrow$ con. $R - \{\text{zeros } P_2(t)\}$

$\sin t$, $\cos t \longrightarrow$ conn. on R.

$\tan t$, $\sec t$, $\csc t$, $\cot t$ - -

↳ $\dfrac{\sin t}{\cos t} \Rightarrow \cos t = 0 \longrightarrow t = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$

cont on $R - \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\}$

$\ln t \longrightarrow$ conti on $(0, \infty)$

Thm

principle of Superposition :-

If y_1, y_2 are solution of the D.E $y'' + p(t)y' + q(t)y = 0$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of c_1, c_2.

Thm Suppose y_1, y_2 are sol. of the DE

$$y'' + p(t)y' + q(t)y = 0$$

and wronskian $= W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

is not zero at the point $t_0 \in I$.

$\Rightarrow$ there is a choice of constants c_1, c_2 for which $y = c_1y_1 + c_2y_2$.

$\Rightarrow$ for the fundamental set of solution $\Rightarrow W(y_1, y_2) \neq 0$ at $t_0 \in I$.

ex show that $y_1(t) = t^{1/2}$, $y_2(t) = t^{-1}$

form a fundamental set of solutions

of $2t^2 y'' + 3t y' - y = 0$, $t > 0$

sol: $\boxed{1}$ show that y_1, y_2 are solutions of the DE :-

(a) y_1 is a sol :-

$$y_1 = t^{1/2} \to y_1' = \frac{1}{2} t^{-1/2} \to y_1'' = -\frac{1}{4} t^{-3/2}$$

$$2t^2 y'' + 3t y' - y = 0$$

$$2t^2 \left(-\frac{1}{4} t^{-3/2}\right) + 3t\left(\frac{1}{2} t^{-1/2}\right) - t^{1/2} = 0$$

$$-\frac{1}{2} t^{1/2} + \frac{3}{2} t^{1/2} - t^{1/2} \overset{??}{=} 0$$

$$-\frac{3}{2} t^{1/2} + \frac{3}{2} t^{1/2} = 0$$

$0 = 0$ ✓ y_1 is a sol. of the D.E

(b) y_2 is a sol :-

$$y_2 = t^{-1} \to y_2' = -t^{-2} \to y_2'' = 2t^{-3}$$

$$2t^2 (2t^{-3}) + 3t(-t)^{-2} - (t^{-1}) = 0$$

$$4t^{-1} - 3t^{-1} - t^{-1} = 0$$

$0 = 0$ ✓ y_2 is a sol.

$\boxed{2}$ show that $W \neq 0$. for $t>0$

$$W(y_1, y_2) = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix}$$

$$= -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2}$$

$$= \frac{-3}{2}\frac{1}{t^{3/2}} \neq 0$$

$W(y_1, y_2) \neq 0$ for $t>0$

$\boxed{1}$ & $\boxed{2}$

$\rightarrow$ then y_1, y_2 form a fundament set of sol. of the D.E.

3.3 Linear Independence and the wronskian.

⊛ functions f, g.

$c_1 f + c_2 g = 0 \Rightarrow c_1, c_2 = 0 \rightarrow f, g$ linearly independ

$\downarrow$

$\exists\, c_1, c_2 \neq 0 \rightarrow$ linearly dependent.

note:

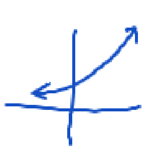
If $W(f,g) \neq 0 \rightarrow f$ and g are linearly independent.

ex $f(t) = e^{2t}$, $g(t) = e^{3t}$ are f, g linearly independent?

sol:

$$W(e^{2t}, e^{3t}) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix}$$

$$= 3e^{5t} - 2e^{5t} = e^{5t} \neq 0$$

(exp. $\neq 0$ )

$\rightarrow f, g$ are linearly independent.

Thm Abel's thm

If y_1, y_2 are solutions for

$$y'' + p(t)\, y' + g(t)\, y = 0$$

then

$$\text{Wronskian} = W(y_1, y_2) = c\, e^{-\int p(t)\, dt}$$

ex $2t^2 y'' + 3t y' - y = 0$.

the solutions are $y_1 = t^{1/2}$

$y_2 = t^{-1}$.

find the wronskian.

<u>sol:</u>

use Abel's thm $\rightarrow$ $1 = y''$ ...

$\div 2t^2 \Rightarrow y'' + \frac{3t}{2t^2} y' - \frac{1}{2t^2} y = 0$

$$y'' + \underbrace{\frac{3}{2t}}_{P(t)} y' - \frac{1}{2t^2} y = 0$$

$$W = c e^{-\int P(t)\,dt}$$

$$= c e^{-\int \frac{3}{2t}\,dt} = c e^{-\frac{3}{2}\int \frac{1}{t}\,dt}$$

$$= c e^{-\frac{3}{2}\ln t} = c e^{\ln t^{-\frac{3}{2}}}$$

$$\boxed{W = c\, t^{-\frac{3}{2}}}$$

$$\boxed{\begin{gathered} e^{\ln x} = x \\ r\ln x = \ln x^r \end{gathered}}$$

another way to find wronskian.

$$w(y_1,y_2) = w(t^{1/2}, t^{-1}) = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix}$$

$$= -t^{-\frac{3}{2}} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-\frac{3}{2}}$$

$$\boxed{W = -\frac{3}{2}t^{-3/2}} \rightarrow \boxed{c = -\frac{3}{2}}$$

Thm $y'' + p(t)y' + q(t)y = 0$

p, q are continuous on I, y_1, y_2 solution of the D.E

y_1, y_2 linearly dependent on I $\iff$ $W(y_1,y_2) = 0$ for all $t \in I$

Thm $y'' + p(t)y' + q(t)y = 0$

p, q cont., y_1, y_2 solutions - -

the following are equivalent:

1. functions y_1, y_2 are fundamental set of solutions on I.
2. y_1, y_2 linearly independent
3. $W(y_1,y_2)(t_0) \neq 0$ for some t_0 in I
4. $W(y_1,y_2)(t) = 0$ for all t in I.