75. Locate and identify the absolute extreme values of

a. $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,

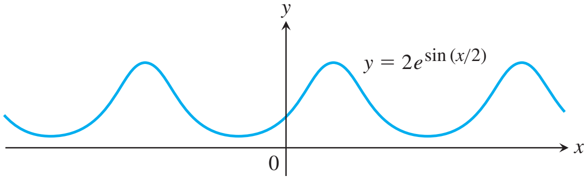
b. $\cos(\ln x)$ on $[1/2, 2]$.

76. a. Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

b. Using part (a), show that $\ln x < x$ if $x > 1$.

77. Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

78. Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



79. Find the absolute maximum value of $f(x) = x^2 \ln(1/x)$ and say where it is assumed.

80. a. Prove that $e^x \geq 1 + x$ if $x \geq 0$.

b. Use the result in part (a) to show that

$$e^x \geq 1 + x + \frac{1}{2}x^2.$$

81. Show that increasing functions and decreasing functions are one-to-one. That is, show that for any x_1 and x_2 in I, $x_2 \neq x_1$ implies $f(x_2) \neq f(x_1)$.

Use the results of Exercise 81 to show that the functions in Exercises 82–86 have inverses over their domains. Find a formula for df^{-1}/dx using Theorem 3, Section 3.8.

82. $f(x) = (1/3)x + (5/6)$

83. $f(x) = 27x^3$

84. $f(x) = 1 - 8x^3$

85. $f(x) = (1 - x)^3$

86. $f(x) = x^{5/3}$

4.4 Concavity and Curve Sketching

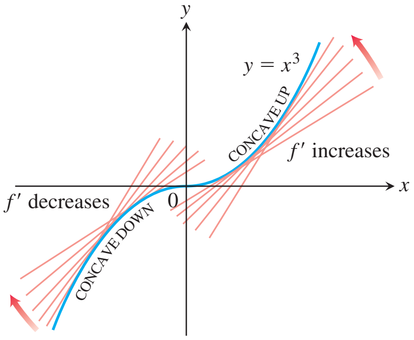


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic behavior studied in Sections 1.1 and 2.6, we can now draw an accurate graph of a function. By organizing all of these ideas into a coherent procedure, we give a method for sketching graphs and revealing visually the key features of functions. Identifying and knowing the locations of these features is of major importance in mathematics and its applications to science and engineering, especially in the graphical analysis and interpretation of data.

Concavity

As you can see in Figure 4.24, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to the first derivative function. We conclude that f' increases if $f'' > 0$ on I, and decreases if $f'' < 0$.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

If $y = f(x)$ is twice-differentiable, we will use the notations f'' and y'' interchangeably when denoting the second derivative.

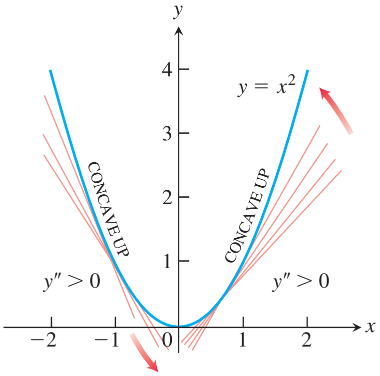


FIGURE 4.25 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

EXAMPLE 1

(a) The curve $y = x^3$ (Figure 4.24) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 4.25) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

EXAMPLE 2

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.26). ■

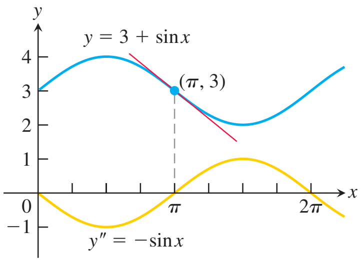


FIGURE 4.26 Using the sign of y'' to determine the concavity of y (Example 2).

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. Since the first derivative $y' = \cos x$ exists for all x, we see that the curve has a tangent line of slope -1 at the point $(\pi, 3)$. This point is called a *point of inflection* of the curve. Notice from Figure 4.26 that the graph crosses its tangent line at this point and that the second derivative $y'' = -\sin x$ has value 0 when $x = \pi$. In general, we have the following definition.

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

We observed that the second derivative of $f(x) = 3 + \sin x$ is equal to zero at the inflection point $(\pi, 3)$. Generally, if the second derivative exists at a point of inflection $(c, f(c))$, then $f''(c) = 0$. This follows immediately from the Intermediate Value Theorem whenever f'' is continuous over an interval containing $x = c$ because the second derivative changes sign moving across this interval. Even if the continuity assumption is dropped, it is still true that $f''(c) = 0$, provided the second derivative exists (although a more advanced argument is required in this noncontinuous case). Since a tangent line must exist at the point of inflection, either the first derivative $f'(c)$ exists (is finite) or the graph has a vertical tangent at the point. At a vertical tangent neither the first nor second derivative exists. In summary, we conclude the following result.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

The next example illustrates a function having a point of inflection where the first derivative exists, but the second derivative fails to exist.

EXAMPLE 3 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.27. ■

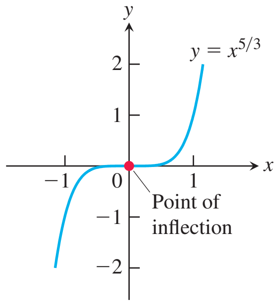


FIGURE 4.27 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 3).

Here is an example showing that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

EXAMPLE 4 The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.28). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. ■

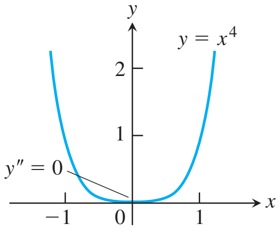


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 4).

As our final illustration, we show a situation in which a point of inflection occurs at a vertical tangent to the curve where neither the first nor the second derivative exists.

EXAMPLE 5 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.29. ■

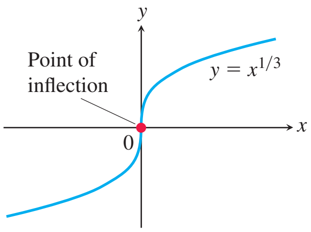


FIGURE 4.29 A point of inflection where y' and y'' fail to exist (Example 5).

Caution Example 4 in Section 4.1 (Figure 4.9) shows that the function $f(x) = x^{2/3}$ does not have a second derivative at $x = 0$ and does not have a point of inflection there (there is no change in concavity at $x = 0$). Combined with the behavior of the function in Example 5 above, we see that when the second derivative does not exist at $x = c$, an inflection point may or may not occur there. So we need to be careful about interpreting functional behavior whenever first or second derivatives fail to exist at a point. At such points the graph can have vertical tangents, corners, cusps, or various discontinuities.

To study the motion of an object moving along a line as a function of time, we often are interested in knowing when the object's acceleration, given by the second derivative, is positive or negative. The points of inflection on the graph of the object's position function reveal where the acceleration changes sign.

EXAMPLE 6 A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \ge 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative $(v = s')$ is zero at the critical points $t = 1$ and $t = 11/3$.

Interval	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest) at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

Interval	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

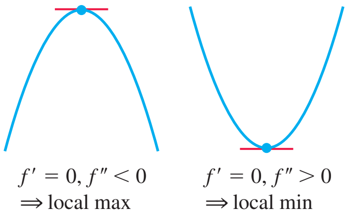
The particle starts out moving to the right while slowing down, and then reverses and begins moving to the left at $t = 1$ under the influence of the leftward acceleration over the time interval $[0, 7/3)$. The acceleration then changes direction at $t = 7/3$ but the particle continues moving leftward, while slowing down under the rightward acceleration. At $t = 11/3$ the particle reverses direction again: moving to the right in the same direction as the acceleration, so it is speeding up. ■

Second Derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and nature of local extrema.

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



Proof Part (1). If $f''(c) < 0$, then $f''(x) < 0$ on some open interval I containing the point c, since f'' is continuous. Therefore, f' is decreasing on I. Since $f'(c) = 0$, the sign of f' changes from positive to negative at c so f has a local maximum at c by the First Derivative Test.

The proof of Part (2) is similar.

For Part (3), consider the three functions $y = x^4$, $y = -x^4$, and $y = x^3$. For each function, the first and second derivatives are zero at $x = 0$. Yet the function $y = x^4$ has a local minimum there, $y = -x^4$ has a local maximum, and $y = x^3$ is increasing in any open interval containing $x = 0$ (having neither a maximum nor a minimum there). Thus the test fails. ■

This test requires us to know f'' *only at c itself* and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $f'' = 0$ or if f'' does not exist at $x = c$. When this happens, use the First Derivative Test for local extreme values.

Together f' and f'' tell us the shape of the function's graph—that is, where the critical points are located and what happens at a critical point, where the function is increasing and where it is decreasing, and how the curve is turning or bending as defined by its concavity. We use this information to sketch a graph of the function that captures its key features.

EXAMPLE 7 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where f is concave up or concave down.

Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

(d) Summarizing the information in the last two tables, we obtain the following.

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

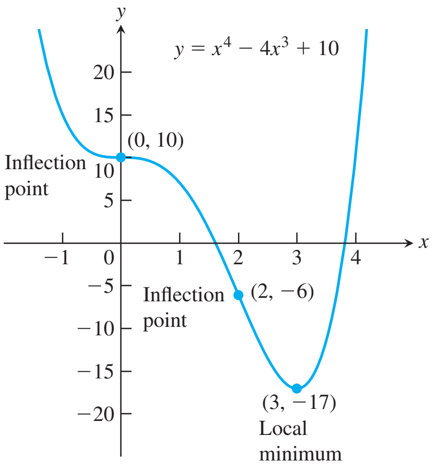
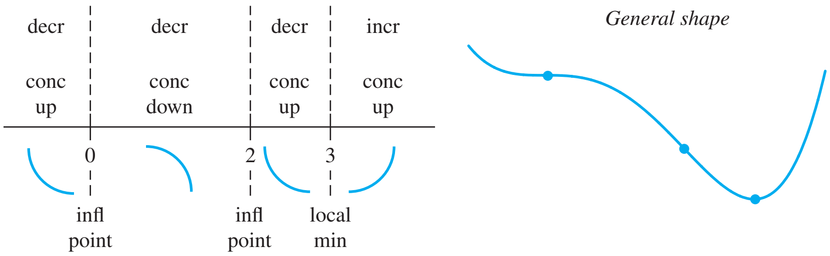


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 7).

The general shape of the curve is shown in the accompanying figure.



(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.30 shows the graph of f. ■

The steps in Example 7 give a procedure for graphing the key features of a function. Asymptotes were defined and discussed in Section 2.6. We can find them for rational functions, and the methods in the next section give tools to help find them for more general functions.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

EXAMPLE 8 Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x-intercept at $x = -1$, y-intercept ($y = 1$) at $x = 0$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2\cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2\cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

After some algebra

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$, yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$, yielding a relative maximum by the Second Derivative test.

4. *Increasing and decreasing*. We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.
5. *Inflection points*. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $\left(-\infty, -\sqrt{3}\right)$, positive on $\left(-\sqrt{3}, 0\right)$, negative on $\left(0, \sqrt{3}\right)$, and positive again on $\left(\sqrt{3}, \infty\right)$. Thus each point is a point of inflection. The curve is concave down on the interval $\left(-\infty, -\sqrt{3}\right)$, concave up on $\left(-\sqrt{3}, 0\right)$, concave down on $\left(0, \sqrt{3}\right)$, and concave up again on $\left(\sqrt{3}, \infty\right)$.
6. *Asymptotes*. Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$
\begin{aligned}
f(x) &= \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} && \text{Expanding numerator} \\
&= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}. && \text{Dividing by } x^2
\end{aligned}
$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \le y \le 2$).

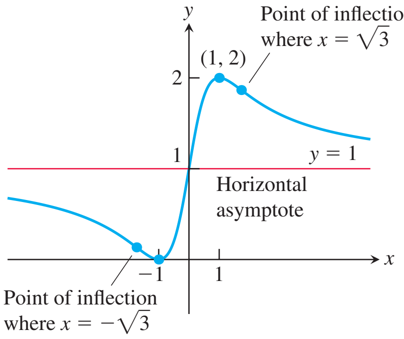


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 8).

7. The graph of f is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■

EXAMPLE 9

Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$.

Solution

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.
2. We calculate the derivatives of the function, but first rewrite it in order to simplify our computations:

$$
\begin{aligned}
f(x) &= \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x} && \text{Function simplified for differentiation} \\
f'(x) &= \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2} && \text{Combine fractions to solve easily } f'(x) = 0. \\
f''(x) &= \frac{4}{x^3} && \text{Exists throughout the entire domain of } f
\end{aligned}
$$

3. The critical points occur at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.

4. On the interval $(-\infty, -2)$ the derivative f' is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.
5. There are no points of inflection because $f''(x) < 0$ whenever $x < 0$, $f''(x) > 0$ whenever $x > 0$, and f'' exists everywhere and is never zero throughout the domain of f. The graph is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.
6. From the rewritten formula for $f(x)$, we see that

$$\lim_{x\to 0^+}\left(\frac{x}{2}+\frac{2}{x}\right) = +\infty \quad\text{and}\quad \lim_{x\to 0^-}\left(\frac{x}{2}+\frac{2}{x}\right) = -\infty,$$

so the y-axis is a vertical asymptote. Also, as $x \to \infty$ or as $x \to -\infty$, the graph of $f(x)$ approaches the line $y = x/2$. Thus $y = x/2$ is an oblique asymptote.
7. The graph of f is sketched in Figure 4.32. ■

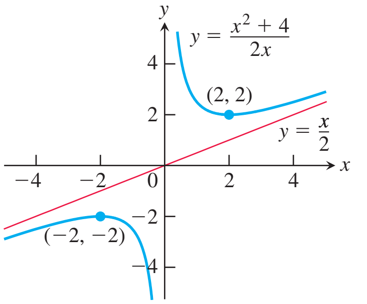


FIGURE 4.32 The graph of $y = \dfrac{x^2+4}{2x}$ (Example 9).

EXAMPLE 10 Sketch the graph of $f(x) = e^{2/x}$.

Solution The domain of f is $(-\infty, 0)\cup(0, \infty)$ and there are no symmetries about either axis or the origin. The derivatives of f are

$$f'(x) = e^{2/x}\left(-\frac{2}{x^2}\right) = -\frac{2e^{2/x}}{x^2}$$

and

$$f''(x) = -\frac{x^2(2e^{2/x})(-2/x^2) - 2e^{2/x}(2x)}{x^4} = \frac{4e^{2/x}(1+x)}{x^4}.$$

Both derivatives exist everywhere over the domain of f. Moreover, since $e^{2/x}$ and x^2 are both positive for all $x \neq 0$, we see that $f' < 0$ everywhere over the domain and the graph is everywhere decreasing. Examining the second derivative, we see that $f''(x) = 0$ at $x = -1$. Since $e^{2/x} > 0$ and $x^4 > 0$, we have $f'' < 0$ for $x < -1$ and $f'' > 0$ for $x > -1, x \neq 0$. Therefore, the point $(-1, e^{-2})$ is a point of inflection. The curve is concave down on the interval $(-\infty, -1)$ and concave up over $(-1, 0) \cup (0, \infty)$.

From Example 7, Section 2.6, we see that $\lim_{x\to 0^-} f(x) = 0$. As $x \to 0^+$, we see that $2/x \to \infty$, so $\lim_{x\to 0^+} f(x) = \infty$ and the y-axis is a vertical asymptote. Also, as $x \to -\infty$ or $x \to \infty$, $2/x \to 0$ and so $\lim_{x\to -\infty} f(x) = \lim_{x\to\infty} f(x) = e^0 = 1$. Therefore, $y = 1$ is a horizontal asymptote. There are no absolute extrema, since f never takes on the value 0 and has no absolute maximum. The graph of f is sketched in Figure 4.33. ■

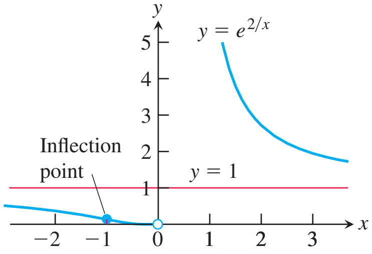
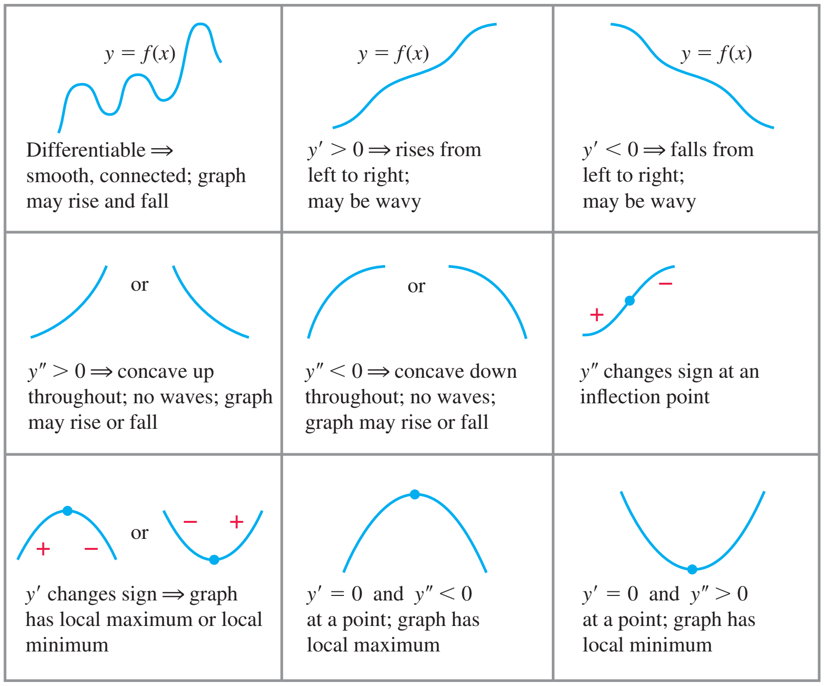


FIGURE 4.33 The graph of $y = e^{2/x}$ has a point of inflection at $(-1, e^{-2})$. The line $y = 1$ is a horizontal asymptote and $x = 0$ is a vertical asymptote (Example 10).

Graphical Behavior of Functions from Derivatives

As we saw in Examples 7–10, we can learn much about a twice-differentiable function $y = f(x)$ by examining its first derivative. We can find where the function's graph rises and falls and where any local extrema are located. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. Information we cannot get from the derivative is how to place the graph in the xy-plane. But, as we discovered in Section 4.2, the only additional information we need to position the graph is the value of f at one point. Information about the asymptotes is found using limits (Section 2.6). The following figure summarizes how the first derivative and second derivative affect the shape of a graph.



Differentiable ⇒ smooth, connected; graph may rise and fall

$y' > 0 \Rightarrow$ rises from left to right; may be wavy

$y' < 0 \Rightarrow$ falls from left to right; may be wavy

$y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall

$y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall

y'' changes sign at an inflection point

y' changes sign ⇒ graph has local maximum or local minimum

$y' = 0$ and $y'' < 0$ at a point; graph has local maximum

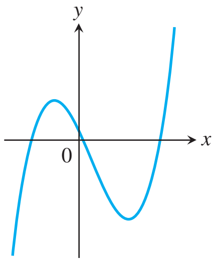
$y' = 0$ and $y'' > 0$ at a point; graph has local minimum

Exercises 4.4

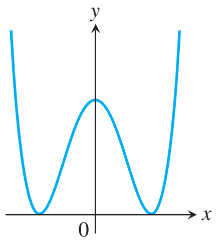
Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

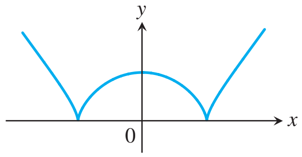
1. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



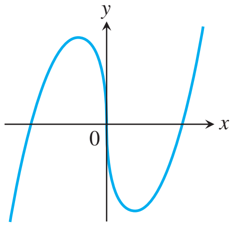
2. $y = \frac{x^4}{4} - 2x^2 + 4$



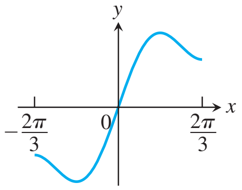
3. $y = \frac{3}{4}(x^2 - 1)^{2/3}$



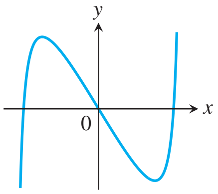
4. $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



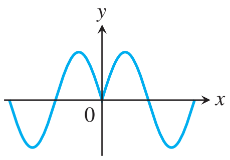
5. $y = x + \sin 2x, \ -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



6. $y = \tan x - 4x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

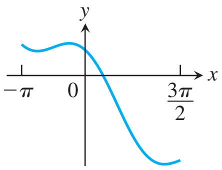


7. $y = \sin|x|, \ -2\pi \le x \le 2\pi$



NOT TO SCALE

8. $y = 2\cos x - \sqrt{2}x, \ -\pi \le x \le \frac{3\pi}{2}$



Graphing Functions

In Exercises 9–58, identify the coordinates of any local and absolute extreme points and inflection points. Graph the function.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$

20. $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$

22. $y = x\left(\frac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$

26. $y = \frac{4}{3}x - \tan x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

27. $y = \sin x \cos x, \quad 0 \le x \le \pi$

28. $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

29. $y = x^{1/5}$

30. $y = x^{2/5}$

31. $y = \frac{x}{\sqrt{x^2 + 1}}$

32. $y = \frac{\sqrt{1 - x^2}}{2x + 1}$

33. $y = 2x - 3x^{2/3}$

34. $y = 5x^{2/5} - 2x$

35. $y = x^{2/3}\left(\frac{5}{2} - x\right)$

36. $y = x^{2/3}(x - 5)$

37. $y = x\sqrt{8 - x^2}$

38. $y = (2 - x^2)^{3/2}$

39. $y = \sqrt{16 - x^2}$

40. $y = x^2 + \frac{2}{x}$

41. $y = \frac{x^2 - 3}{x - 2}$

42. $y = \sqrt[3]{x^3 + 1}$

43. $y = \frac{8x}{x^2 + 4}$

44. $y = \frac{5}{x^4 + 5}$

45. $y = |x^2 - 1|$

46. $y = |x^2 - 2x|$

47. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

48. $y = \sqrt{|x - 4|}$

49. $y = xe^{1/x}$

50. $y = \frac{e^x}{x}$

51. $y = \ln(3 - x^2)$

52. $y = x(\ln x)^2$

53. $y = e^x - 2e^{-x} - 3x$

54. $y = xe^{-x}$

55. $y = \ln(\cos x)$

56. $y = \frac{\ln x}{\sqrt{x}}$

57. $y = \frac{1}{1 + e^{-x}}$

58. $y = \frac{e^x}{1 + e^x}$

Sketching the General Shape, Knowing y'

Each of Exercises 59–80 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use Steps 2–4 of the graphing procedure on page 249 to sketch the general shape of the graph of f.

59. $y' = 2 + x - x^2$

60. $y' = x^2 - x - 6$

61. $y' = x(x - 3)^2$

62. $y' = x^2(2 - x)$

63. $y' = x(x^2 - 12)$

64. $y' = (x - 1)^2(2x + 3)$

65. $y' = (8x - 5x^2)(4 - x)^2$

66. $y' = (x^2 - 2x)(x - 5)^2$

67. $y' = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

68. $y' = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

69. $y' = \cot\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

70. $y' = \csc^2\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

71. $y' = \tan^2\theta - 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

72. $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

73. $y' = \cos t, \quad 0 \le t \le 2\pi$

74. $y' = \sin t, \quad 0 \le t \le 2\pi$

75. $y' = (x + 1)^{-2/3}$

76. $y' = (x - 2)^{-1/3}$

77. $y' = x^{-2/3}(x - 1)$

78. $y' = x^{-4/5}(x + 1)$

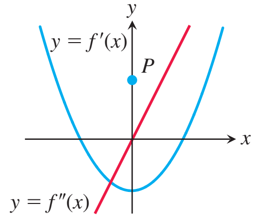
79. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

80. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

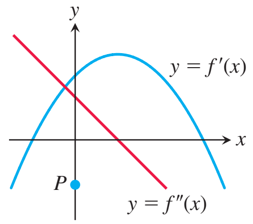
Sketching y from Graphs of y' and y''

Each of Exercises 81–84 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

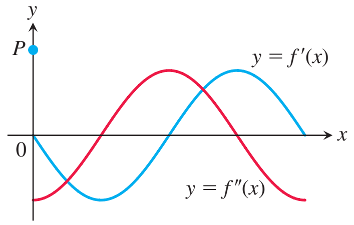
81.



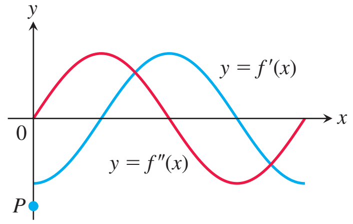
82.



83.



84.



Graphing Rational Functions

Graph the rational functions in Exercises 85–102 using all the steps in the graphing procedure on page 249.

85. $y = \frac{2x^2 + x - 1}{x^2 - 1}$

86. $y = \frac{x^2 - 49}{x^2 + 5x - 14}$

87. $y = \frac{x^4 + 1}{x^2}$

88. $y = \frac{x^2 - 4}{2x}$

89. $y = \frac{1}{x^2 - 1}$

90. $y = \frac{x^2}{x^2 - 1}$

91. $y = -\dfrac{x^2-2}{x^2-1}$

92. $y = \dfrac{x^2-4}{x^2-2}$

93. $y = \dfrac{x^2}{x+1}$

94. $y = -\dfrac{x^2-4}{x+1}$

95. $y = \dfrac{x^2-x+1}{x-1}$

96. $y = -\dfrac{x^2-x+1}{x-1}$

97. $y = \dfrac{x^3-3x^2+3x-1}{x^2+x-2}$

98. $y = \dfrac{x^3+x-2}{x-x^2}$

99. $y = \dfrac{x}{x^2-1}$

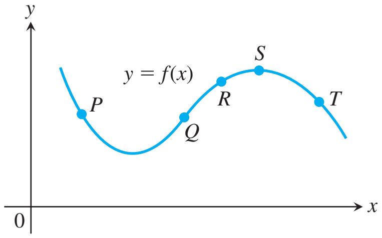
100. $y = \dfrac{x-1}{x^2(x-2)}$

101. $y = \dfrac{8}{x^2+4}$ (Agnesi's witch)

102. $y = \dfrac{4x}{x^2+4}$ (Newton's serpentine)

Theory and Examples

103. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



104. Sketch a smooth connected curve $y = f(x)$ with

$f(-2) = 8,$
$f(0) = 4,$
$f(2) = 0,$
$f'(x) > 0$ for $|x| > 2,$
$f'(2) = f'(-2) = 0,$
$f'(x) < 0$ for $|x| < 2,$
$f''(x) < 0$ for $x < 0,$
$f''(x) > 0$ for $x > 0.$

105. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	**Derivatives**
$x < 2$		$y' < 0,\quad y'' > 0$
2	1	$y' = 0,\quad y'' > 0$
$2 < x < 4$		$y' > 0,\quad y'' > 0$
4	4	$y' > 0,\quad y'' = 0$
$4 < x < 6$		$y' > 0,\quad y'' < 0$
6	7	$y' = 0,\quad y'' < 0$
$x > 6$		$y' < 0,\quad y'' < 0$

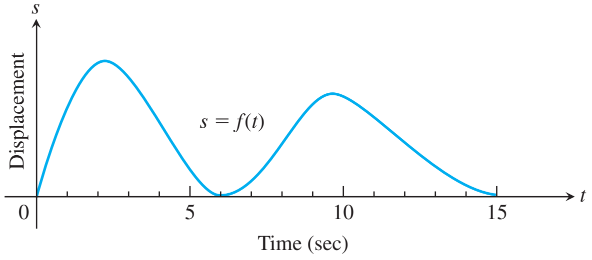
106. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 2)$ and whose first two derivatives have the following sign patterns.

$$y':\ \begin{array}{ccccccc} + & & - & & + & & - \\ \hline & -2 & & 0 & & 2 & \end{array}$$

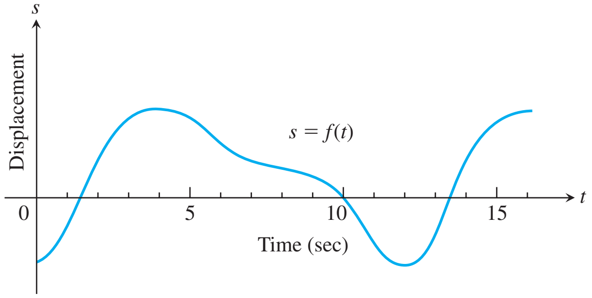
$$y'':\ \begin{array}{ccccc} - & & + & & - \\ \hline & -1 & & 1 & \end{array}$$

Motion Along a Line The graphs in Exercises 107 and 108 show the position $s = f(t)$ of an object moving up and down on a coordinate line. **(a)** When is the object moving away from the origin? Toward the origin? At approximately what times is the **(b)** velocity equal to zero? **(c)** Acceleration equal to zero? **(d)** When is the acceleration positive? Negative?

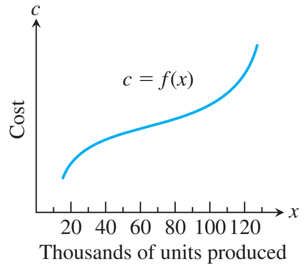
107.



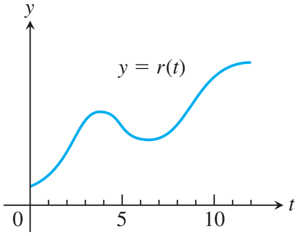
108.



109. Marginal cost The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing x items. At approximately what production level does the marginal cost change from decreasing to increasing?



110. The accompanying graph shows the monthly revenue of the Widget Corporation for the past 12 years. During approximately what time intervals was the marginal revenue increasing? Decreasing?



111. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x-1)^2(x-2).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for y'.)

112. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

113. For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

114. Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

115. If b, c, and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

116. Parabolas

a. Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

b. When is the parabola concave up? Concave down? Give reasons for your answers.

117. Quadratic curves What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

118. Cubic curves What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

119. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x + 1)(x - 2).$$

For what x-values does the graph of f have an inflection point?

120. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x - 2)^3(x + 3).$$

For what x-values does the graph of f have an inflection point?

121. Find the values of constants a, b, and c so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

122. Find the values of constants a, b, and c so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

COMPUTER EXPLORATIONS

In Exercises 123–126, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the x-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

123. $y = x^5 - 5x^4 - 240$ **124.** $y = x^3 - 12x^2$

125. $y = \frac{4}{5}x^5 + 16x^2 - 25$

126. $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

127. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f''.

128. Graph $f(x) = x \cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f''.

4.5 Indeterminate Forms and L'Hôpital's Rule

John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of the rule for their calculation.

HISTORICAL BIOGRAPHY

Guillaume François Antoine de l'Hôpital
(1661–1704)

Johann Bernoulli
(1667–1748)

Indeterminate Form 0/0

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.