112. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

113. For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

114. Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

115. If b, c, and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

116. Parabolas

a. Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

b. When is the parabola concave up? Concave down? Give reasons for your answers.

117. Quadratic curves What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

118. Cubic curves What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

119. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x + 1)(x - 2).$$

For what x-values does the graph of f have an inflection point?

120. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x - 2)^3(x + 3).$$

For what x-values does the graph of f have an inflection point?

121. Find the values of constants a, b, and c so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

122. Find the values of constants a, b, and c so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

COMPUTER EXPLORATIONS

In Exercises 123–126, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the x-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

123. $y = x^5 - 5x^4 - 240$

124. $y = x^3 - 12x^2$

125. $y = \frac{4}{5}x^5 + 16x^2 - 25$

126. $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

127. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f''.

128. Graph $f(x) = x \cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f''.

4.5 Indeterminate Forms and L'Hôpital's Rule

HISTORICAL BIOGRAPHY

Guillaume François Antoine de l'Hôpital (1661–1704)

Johann Bernoulli (1667–1748)

John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of the rule for their calculation.

Indeterminate Form 0/0

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0, and 1^∞, which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic manipulations. This was our experience in Chapter 2. It took considerable analysis in Section 2.4 to find $\lim_{x \to 0} (\sin x)/x$. But we have had success with the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

from which we calculate derivatives and which produces the indeterminant form $0/0$ when we attempt to substitute $x = a$. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

THEOREM 6—L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

We give a proof of Theorem 6 at the end of this section.

Caution
To apply l'Hôpital's Rule to f/g, divide the derivative of f by the derivative of g. Do not fall into the trap of taking the derivative of f/g. The quotient to use is f'/g', not $(f/g)'$.

EXAMPLE 1 The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $$\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = \left. \frac{3 - \cos x}{1} \right|_{x=0} = 2$$

(b) $$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1 + x}}}{1} = \frac{1}{2}$$

(c) $$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2}$$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$= \lim_{x \to 0} \frac{(1/2)(1 + x)^{-1/2} - 1/2}{2x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x \to 0} \frac{-(1/4)(1 + x)^{-3/2}}{2} = -\frac{1}{8}$$ Not $\frac{0}{0}$; limit is found.

(d) $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x \to 0} \frac{\sin x}{6x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}$$ Not $\frac{0}{0}$; limit is found.

Here is a summary of the procedure we followed in Example 1.

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

EXAMPLE 2 Be careful to apply l'Hôpital's Rule correctly:

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$$ $\frac{0}{0}$

$$= \lim_{x \to 0} \frac{\sin x}{1 + 2x}$$ Not $\frac{0}{0}$

It is tempting to try to apply l'Hôpital's Rule again, which would result in

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2},$$

but this is not the correct limit. L'Hôpital's Rule can be applied only to limits that give indeterminate forms, and $\lim_{x \to 0} (\sin x)/(1 + 2x)$ does not give an indeterminate form. Instead, this limit is $0/1 = 0$, and the correct answer for the original limit is 0.

L'Hôpital's Rule applies to one-sided limits as well.

EXAMPLE 3 In this example the one-sided limits are different.

(a) $\lim_{x \to 0^+} \frac{\sin x}{x^2}$ $\frac{0}{0}$

$$= \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty$$ Positive for $x > 0$

Recall that ∞ and $+\infty$ mean the same thing.

(b) $\lim_{x \to 0^-} \frac{\sin x}{x^2}$ $\frac{0}{0}$

$$= \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty$$ Negative for $x < 0$

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminant form like ∞/∞, $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. We first consider the form ∞/∞.

More advanced treatments of calculus prove that l'Hôpital's Rule applies to the indeterminate form ∞/∞, as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x \to a$, a may be either finite or infinite. Moreover, $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

EXAMPLE 4 Find the limits of these ∞/∞ forms:

(a) $\lim_{x\to\pi/2} \frac{\sec x}{1 + \tan x}$ **(b)** $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty} \text{ from the left so we apply l'Hôpital's Rule.}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \qquad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

(c) $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$ ■

Next we turn our attention to the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$. Sometimes these forms can be handled by using algebra to convert them to a $0/0$ or ∞/∞ form. Here again we do not mean to suggest that $\infty \cdot 0$ or $\infty - \infty$ is a number. They are only notations for functional behaviors when considering limits. Here are examples of how we might work with these indeterminate forms.

EXAMPLE 5 Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x\to\infty}\left(x \sin \frac{1}{x}\right)$ **(b)** $\lim_{x\to 0^+} \sqrt{x} \ln x$

Solution

a. $$\lim_{x\to\infty}\left(x \sin \frac{1}{x}\right) = \lim_{h\to 0^+}\left(\frac{1}{h} \sin h\right) = \lim_{h\to 0^+} \frac{\sin h}{h} = 1 \qquad \infty \cdot 0; \text{ let } h = 1/x.$$

b. $$\lim_{x\to 0^+} \sqrt{x} \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} \qquad \infty \cdot 0 \text{ converted to } \infty/\infty$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/2x^{3/2}} \qquad \text{l'Hôpital's Rule applied}$$

$$= \lim_{x\to 0^+} \left(-2\sqrt{x}\right) = 0$$ ■

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}. \qquad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x\to 0}\frac{x - \sin x}{x \sin x} && \frac{0}{0} \\
&= \lim_{x\to 0}\frac{1 - \cos x}{\sin x + x\cos x} && \text{Still } \frac{0}{0} \\
&= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.
\end{aligned}$$

■

Indeterminate Powers

Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by first taking the logarithm of the function. We use l'Hôpital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit. This procedure is justified by the continuity of the exponential function and Theorem 10 in Section 2.5, and it is formulated as follows. (The formula is also valid for one-sided limits.)

> If $\lim_{x\to a} \ln f(x) = L$, then
>
> $$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$
>
> Here a may be either finite or infinite.

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x\to 0^+}(1 + x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x\to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln (1 + x)^{1/x} = \frac{1}{x}\ln(1 + x),$$

l'Hôpital's Rule now applies to give

$$\begin{aligned}
\lim_{x\to 0^+} \ln f(x) &= \lim_{x\to 0^+}\frac{\ln(1 + x)}{x} && \frac{0}{0} \\
&= \lim_{x\to 0^+}\frac{\frac{1}{1 + x}}{1} && \text{l'Hôpital's Rule applied} \\
&= \frac{1}{1} = 1.
\end{aligned}$$

Therefore, $\lim_{x\to 0^+}(1 + x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e.$ ■

EXAMPLE 8 Find $\lim_{x\to\infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\begin{aligned}\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} && \frac{\infty}{\infty}\\ &= \lim_{x\to\infty} \frac{1/x}{1} && \text{l'Hôpital's Rule applied}\\ &= \frac{0}{1} = 0.\end{aligned}$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$ ■

Proof of L'Hôpital's Rule

Before we prove l'Hôpital's Rule, we consider a special case to provide some geometric insight for its reasonableness. Consider the two functions $f(x)$ and $g(x)$ having *continuous* derivatives and satisfying $f(a) = g(a) = 0$, $g'(a) \neq 0$. The graphs of $f(x)$ and $g(x)$, together with their linearizations $y = f'(a)(x - a)$ and $y = g'(a)(x - a)$, are shown in Figure 4.34. We know that near $x = a$, the linearizations provide good approximations to the functions. In fact,

$$f(x) = f'(a)(x - a) + \epsilon_1(x - a) \quad \text{and} \quad g(x) = g'(a)(x - a) + \epsilon_2(x - a)$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $x \to a$. So, as Figure 4.34 suggests,

$$\begin{aligned}\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f'(a)(x - a) + \epsilon_1(x - a)}{g'(a)(x - a) + \epsilon_2(x - a)}\\ &= \lim_{x\to a} \frac{f'(a) + \epsilon_1}{g'(a) + \epsilon_2} = \frac{f'(a)}{g'(a)} && g'(a) \neq 0\\ &= \lim_{x\to a} \frac{f'(x)}{g'(x)}, && \text{Continuous derivatives}\end{aligned}$$

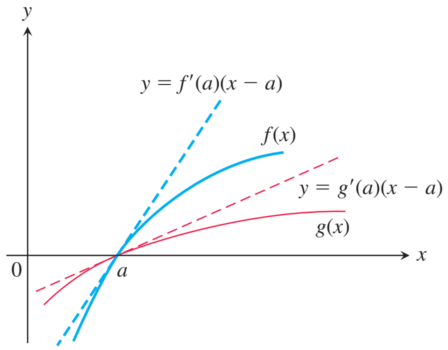

FIGURE 4.34 The two functions in l'Hôpital's Rule, graphed with their linear approximations at $x = a$.

as asserted by l'Hôpital's Rule. We now proceed to a proof of the rule based on the more general assumptions stated in Theorem 6, which do not require that $g'(a) \neq 0$ and that the two functions have *continuous* derivatives.

The proof of l'Hôpital's Rule is based on Cauchy's Mean Value Theorem, an extension of the Mean Value Theorem that involves two functions instead of one. We prove Cauchy's Theorem first and then show how it leads to l'Hôpital's Rule.

HISTORICAL BIOGRAPHY
Augustin-Louis Cauchy
(1789–1857)

THEOREM 7—Cauchy's Mean Value Theorem Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

When $g(x) = x$, Theorem 7 is the Mean Value Theorem.

Proof We apply the Mean Value Theorem of Section 4.2 twice. First we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the Mean Value Theorem would give

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some c between a and b, which cannot happen because $g'(x) \neq 0$ in (a, b).

We next apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)].$$

This function is continuous and differentiable where f and g are, and $F(b) = F(a) = 0$. Therefore, there is a number c between a and b for which $F'(c) = 0$. When expressed in terms of f and g, this equation becomes

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}[g'(c)] = 0$$

so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \blacksquare$$

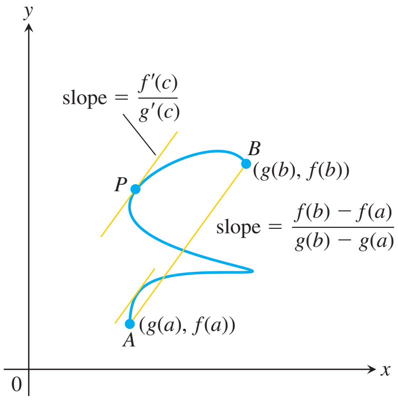

FIGURE 4.35 There is at least one point P on the curve C for which the slope of the tangent to the curve at P is the same as the slope of the secant line joining the points $A(g(a), f(a))$ and $B(g(b), f(b))$.

Cauchy's Mean Value Theorem has a geometric interpretation for a general winding curve C in the plane joining the two points $A = (g(a), f(a))$ and $B = (g(b), f(b))$. In Chapter 11 you will learn how the curve C can be formulated so that there is at least one point P on the curve for which the tangent to the curve at P is parallel to the secant line joining the points A and B. The slope of that tangent line turns out to be the quotient f'/g' evaluated at the number c in the interval (a, b), which is the left-hand side of the equation in Theorem 7. Because the slope of the secant line joining A and B is

$$\frac{f(b) - f(a)}{g(b) - g(a)},$$

the equation in Cauchy's Mean Value Theorem says that the slope of the tangent line equals the slope of the secant line. This geometric interpretation is shown in Figure 4.35. Notice from the figure that it is possible for more than one point on the curve C to have a tangent line that is parallel to the secant line joining A and B.

Proof of l'Hôpital's Rule We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches a, c approaches a because it always lies between a and x. Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

which establishes l'Hôpital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a]$, $x < a$. $\blacksquare$

Exercises 4.5

Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x\to-2}\frac{x+2}{x^2-4}$

2. $\lim_{x\to0}\frac{\sin 5x}{x}$

3. $\lim_{x\to\infty}\frac{5x^2-3x}{7x^2+1}$

4. $\lim_{x\to1}\frac{x^3-1}{4x^3-x-3}$

5. $\lim_{x\to0}\frac{1-\cos x}{x^2}$

6. $\lim_{x\to\infty}\frac{2x^2+3x}{x^3+x+1}$

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x\to2}\frac{x-2}{x^2-4}$

8. $\lim_{x\to-5}\frac{x^2-25}{x+5}$

9. $\lim_{t\to-3}\frac{t^3-4t+15}{t^2-t-12}$

10. $\lim_{t\to-1}\frac{3t^3+3}{4t^3-t+3}$

11. $\lim_{x\to\infty}\frac{5x^3-2x}{7x^3+3}$

12. $\lim_{x\to\infty}\frac{x-8x^2}{12x^2+5x}$

13. $\lim_{t\to0}\frac{\sin t^2}{t}$

14. $\lim_{t\to0}\frac{\sin 5t}{2t}$

15. $\lim_{x\to0}\frac{8x^2}{\cos x-1}$

16. $\lim_{x\to0}\frac{\sin x-x}{x^3}$

17. $\lim_{\theta\to\pi/2}\frac{2\theta-\pi}{\cos(2\pi-\theta)}$

18. $\lim_{\theta\to-\pi/3}\frac{3\theta+\pi}{\sin(\theta+(\pi/3))}$

19. $\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{1+\cos 2\theta}$

20. $\lim_{x\to1}\frac{x-1}{\ln x-\sin\pi x}$

21. $\lim_{x\to0}\frac{x^2}{\ln(\sec x)}$

22. $\lim_{x\to\pi/2}\frac{\ln(\csc x)}{(x-(\pi/2))^2}$

23. $\lim_{t\to0}\frac{t(1-\cos t)}{t-\sin t}$

24. $\lim_{t\to0}\frac{t\sin t}{1-\cos t}$

25. $\lim_{x\to(\pi/2)^-}\left(x-\frac{\pi}{2}\right)\sec x$

26. $\lim_{x\to(\pi/2)^-}\left(\frac{\pi}{2}-x\right)\tan x$

27. $\lim_{\theta\to0}\frac{3^{\sin\theta}-1}{\theta}$

28. $\lim_{\theta\to0}\frac{(1/2)^\theta-1}{\theta}$

29. $\lim_{x\to0}\frac{x2^x}{2^x-1}$

30. $\lim_{x\to0}\frac{3^x-1}{2^x-1}$

31. $\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}$

32. $\lim_{x\to\infty}\frac{\log_2 x}{\log_3(x+3)}$

33. $\lim_{x\to0^+}\frac{\ln(x^2+2x)}{\ln x}$

34. $\lim_{x\to0^+}\frac{\ln(e^x-1)}{\ln x}$

35. $\lim_{y\to0}\frac{\sqrt{5y+25}-5}{y}$

36. $\lim_{y\to0}\frac{\sqrt{ay+a^2}-a}{y},\quad a>0$

37. $\lim_{x\to\infty}(\ln 2x-\ln(x+1))$

38. $\lim_{x\to0^+}(\ln x-\ln\sin x)$

39. $\lim_{x\to0^+}\frac{(\ln x)^2}{\ln(\sin x)}$

40. $\lim_{x\to0^+}\left(\frac{3x+1}{x}-\frac{1}{\sin x}\right)$

41. $\lim_{x\to1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$

42. $\lim_{x\to0^+}(\csc x-\cot x+\cos x)$

43. $\lim_{\theta\to0}\frac{\cos\theta-1}{e^\theta-\theta-1}$

44. $\lim_{h\to0}\frac{e^h-(1+h)}{h^2}$

45. $\lim_{t\to\infty}\frac{e^t+t^2}{e^t-t}$

46. $\lim_{x\to\infty}x^2e^{-x}$

47. $\lim_{x\to0}\frac{x-\sin x}{x\tan x}$

48. $\lim_{x\to0}\frac{(e^x-1)^2}{x\sin x}$

49. $\lim_{\theta\to0}\frac{\theta-\sin\theta\cos\theta}{\tan\theta-\theta}$

50. $\lim_{x\to0}\frac{\sin 3x-3x+x^2}{\sin x\sin 2x}$

Indeterminate Powers and Products

Find the limits in Exercises 51–66.

51. $\lim_{x\to1^+}x^{1/(1-x)}$

52. $\lim_{x\to1^+}x^{1/(x-1)}$

53. $\lim_{x\to\infty}(\ln x)^{1/x}$

54. $\lim_{x\to e^+}(\ln x)^{1/(x-e)}$

55. $\lim_{x\to0^+}x^{-1/\ln x}$

56. $\lim_{x\to\infty}x^{1/\ln x}$

57. $\lim_{x\to\infty}(1+2x)^{1/(2\ln x)}$

58. $\lim_{x\to0}(e^x+x)^{1/x}$

59. $\lim_{x\to0^+}x^x$

60. $\lim_{x\to0^+}\left(1+\frac{1}{x}\right)^x$

61. $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x$

62. $\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{1/x}$

63. $\lim_{x\to0^+}x^2\ln x$

64. $\lim_{x\to0^+}x(\ln x)^2$

65. $\lim_{x\to0^+}x\tan\left(\frac{\pi}{2}-x\right)$

66. $\lim_{x\to0^+}\sin x\cdot\ln x$

Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 67–74. Try it—you just keep on cycling. Find the limits some other way.

67. $\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}$

68. $\lim_{x\to0^+}\frac{\sqrt{x}}{\sqrt{\sin x}}$

69. $\lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan x}$

70. $\lim_{x\to0^+}\frac{\cot x}{\csc x}$

71. $\lim_{x\to\infty}\frac{2^x-3^x}{3^x+4^x}$

72. $\lim_{x\to-\infty}\frac{2^x+4^x}{5^x-2^x}$

73. $\lim_{x\to\infty}\frac{e^{x^2}}{xe^x}$

74. $\lim_{x\to0^+}\frac{x}{e^{-1/x}}$

75. Which one is correct, and which one is wrong? Give reasons for your answers.

a. $\lim_{x\to3}\frac{x-3}{x^2-3}=\lim_{x\to3}\frac{1}{2x}=\frac{1}{6}$ **b.** $\lim_{x\to3}\frac{x-3}{x^2-3}=\frac{0}{6}=0$

76. Which one is correct, and which one is wrong? Give reasons for your answers.

a. $\lim_{x\to0}\frac{x^2-2x}{x^2-\sin x}=\lim_{x\to0}\frac{2x-2}{2x-\cos x}$

$=\lim_{x\to0}\frac{2}{2+\sin x}=\frac{2}{2+0}=1$

b. $\lim_{x\to0}\frac{x^2-2x}{x^2-\sin x}=\lim_{x\to0}\frac{2x-2}{2x-\cos x}=\frac{-2}{0-1}=2$

77. Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

a. $\lim_{x\to 0^+} x \ln x = 0 \cdot (-\infty) = 0$

b. $\lim_{x\to 0^+} x \ln x = 0 \cdot (-\infty) = -\infty$

c. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{(1/x)} = \frac{-\infty}{\infty} = -1$

d. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{(1/x)}$

$$= \lim_{x\to 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+} (-x) = 0$$

78. Find all values of c that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

a. $f(x) = x, \quad g(x) = x^2, \quad (a, b) = (-2, 0)$

b. $f(x) = x, \quad g(x) = x^2, \quad (a, b)$ arbitrary

c. $f(x) = x^3/3 - 4x, \quad g(x) = x^2, \quad (a, b) = (0, 3)$

79. Continuous extension Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

80. For what values of a and b is

$$\lim_{x\to 0} \left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x} \right) = 0?$$

T **81. ∞ − ∞ Form**

a. Estimate the value of

$$\lim_{x\to\infty} \left(x - \sqrt{x^2 + x} \right)$$

by graphing $f(x) = x - \sqrt{x^2 + x}$ over a suitably large interval of x-values.

b. Now confirm your estimate by finding the limit with l'Hôpital's Rule. As the first step, multiply $f(x)$ by the fraction $\left(x + \sqrt{x^2 + x}\right)/\left(x + \sqrt{x^2 + x}\right)$ and simplify the new numerator.

82. Find $\lim_{x\to\infty} \left(\sqrt{x^2 + 1} - \sqrt{x} \right)$.

T **83. 0/0 Form** Estimate the value of

$$\lim_{x\to 1} \frac{2x^2 - (3x + 1)\sqrt{x} + 2}{x - 1}$$

by graphing. Then confirm your estimate with l'Hôpital's Rule.

84. This exercise explores the difference between the limit

$$\lim_{x\to\infty} \left(1 + \frac{1}{x^2}\right)^x$$

and the limit

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

a. Use l'Hôpital's Rule to show that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

T **b.** Graph

$$f(x) = \left(1 + \frac{1}{x^2}\right)^x \quad \text{and} \quad g(x) = \left(1 + \frac{1}{x}\right)^x$$

together for $x \geq 0$. How does the behavior of f compare with that of g? Estimate the value of $\lim_{x\to\infty} f(x)$.

c. Confirm your estimate of $\lim_{x\to\infty} f(x)$ by calculating it with l'Hôpital's Rule.

85. Show that

$$\lim_{k\to\infty} \left(1 + \frac{r}{k}\right)^k = e^r.$$

86. Given that $x > 0$, find the maximum value, if any, of

a. $x^{1/x}$

b. x^{1/x^2}

c. x^{1/x^n} (n a positive integer)

d. Show that $\lim_{x\to\infty} x^{1/x^n} = 1$ for every positive integer n.

87. Use limits to find horizontal asymptotes for each function.

a. $y = x \tan\left(\frac{1}{x}\right)$ **b.** $y = \frac{3x + e^{2x}}{2x + e^{3x}}$

88. Find $f'(0)$ for $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

T **89. The continuous extension of $(\sin x)^x$ to $[0, \pi]$**

a. Graph $f(x) = (\sin x)^x$ on the interval $0 \leq x \leq \pi$. What value would you assign to f to make it continuous at $x = 0$?

b. Verify your conclusion in part (a) by finding $\lim_{x\to 0^+} f(x)$ with l'Hôpital's Rule.

c. Returning to the graph, estimate the maximum value of f on $[0, \pi]$. About where is max f taken on?

d. Sharpen your estimate in part (c) by graphing f' in the same window to see where its graph crosses the x-axis. To simplify your work, you might want to delete the exponential factor from the expression for f' and graph just the factor that has a zero.

T **90. The function $(\sin x)^{\tan x}$** (*Continuation of Exercise 89.*)

a. Graph $f(x) = (\sin x)^{\tan x}$ on the interval $-7 \leq x \leq 7$. How do you account for the gaps in the graph? How wide are the gaps?

b. Now graph f on the interval $0 \leq x \leq \pi$. The function is not defined at $x = \pi/2$, but the graph has no break at this point. What is going on? What value does the graph appear to give for f at $x = \pi/2$? (*Hint:* Use l'Hôpital's Rule to find lim f as $x \to (\pi/2)^-$ and $x \to (\pi/2)^+$.)

c. Continuing with the graphs in part (b), find max f and min f as accurately as you can and estimate the values of x at which they are taken on.

4.6 Applied Optimization

What are the dimensions of a rectangle with fixed perimeter having *maximum area*? What are the dimensions for the *least expensive* cylindrical can of a given volume? How many items should be produced for the *most profitable* production run? Each of these questions asks for the best, or optimal, value of a given function. In this section we use derivatives to solve a variety of optimization problems in mathematics, physics, economics, and business.

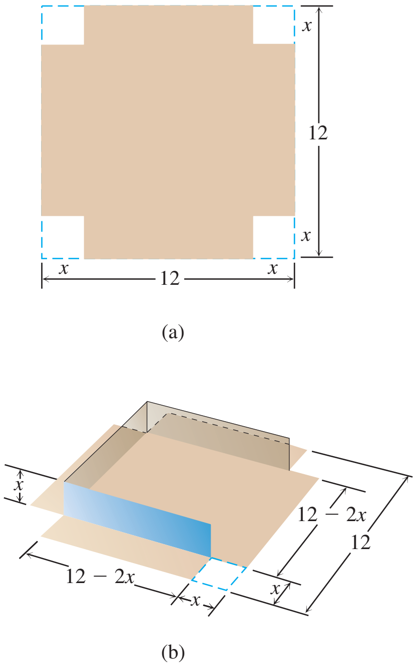

FIGURE 4.36 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

EXAMPLE 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution We start with a picture (Figure 4.36). In the figure, the corner squares are x in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3. \qquad V = hlw$$

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of V is the interval $0 \le x \le 6$.

A graph of V (Figure 4.37) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x:

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

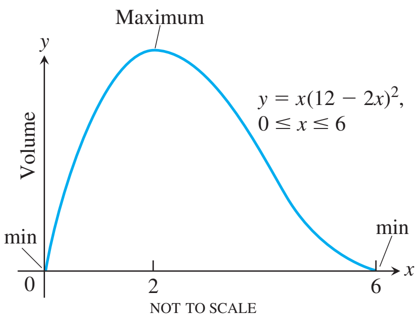

FIGURE 4.37 The volume of the box in Figure 4.36 graphed as a function of x.

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

Critical point value: $V(2) = 128$

Endpoint values: $V(0) = 0, \quad V(6) = 0.$

The maximum volume is 128 in^3. The cutout squares should be 2 in. on a side. ■