42. ***Advertising and sales*** Suppose that sales (in thousands of dollars) are directly related to an advertising campaign according to

$$S = 1 + \frac{3t - 9}{(t + 3)^2}$$

where t is the number of weeks of the campaign.
(a) Find the rate of change of sales after 3 weeks.
(b) Interpret the result in part (a).

43. ***Advertising and sales*** An inferior product with an extensive advertising campaign does well when it is released, but sales decline as people discontinue use of the product. If the sales S (in thousands of dollars) after t weeks are given by

$$S(t) = \frac{200t}{(t + 1)^2}, \quad t \geq 0$$

what is the rate of change of sales when $t = 9$? Interpret your result.

44. ***Advertising and sales*** An excellent film with a very small advertising budget must depend largely on word-of-mouth advertising. If attendance at the film after t weeks is given by

$$A = \frac{100t}{(t + 10)^2}$$

what is the rate of change in attendance and what does it mean when (a) $t = 10$? (b) $t = 20$?

45. ***Union participation*** The following table shows the percent of U.S. workers in unions for selected years from 1930 to 2005.

Year	Percent	Year	Percent
1930	11.6	1975	25.5
1935	13.2	1980	21.9
1940	26.9	1985	18.0
1945	35.5	1990	16.1
1950	31.5	1995	14.9
1955	33.2	2000	13.5
1960	31.4	2002	13.3
1965	28.4	2004	12.5
1970	27.3	2005	12.5

Source: Bureau of Labor Statistics

Assume these data can be modeled with the function

$$U(t) = \frac{9.365(0.2t - 6)^3 - 387.9(0.2t - 6)^2 + 924.2t - 25{,}963}{14.02t - 238.0}$$

where $U(t)$ is the percent of U.S. workers in unions and t is the number of years past 1900.
(a) Find the instantaneous rates of change of the percent in 1935 and 2005.
(b) Interpret the two rates found in part (a).

9.8 Higher-Order Derivatives

OBJECTIVE

- To find second derivatives and higher-order derivatives of certain functions

Application Preview

Since cell phones were introduced, their popularity has increased enormously. Figure 9.33(a) shows a graph of the millions of worldwide cellular subscribers (actual and projected) as a function of the number of years past 1995. (*Source:* International Telecommunications Union) Note that the number of subscribers is always increasing and that the rate of change of that number (as seen from

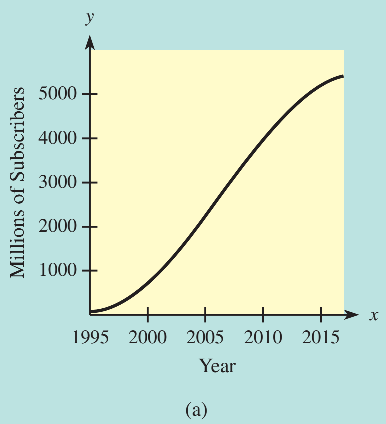

Figure 9.33

tangent lines to the graph) is always positive. However, the tangent lines shown in Figure 9.33(b) indicate that the rate of change of the number of subscribers is greater at *B* than at either *A* or *C*.

Furthermore, the rate of change of the number of subscribers (the slopes of tangents) increases from *A* to *B* and then decreases from *B* to *C*. To learn how the rate of change of the number of subscribers is changing, we are interested in finding the derivative of the rate of change of the number of subscribers—that is, the derivative of the derivative of the number of subscribers. (See Example 4.) This is called the **second derivative.** In this section we will discuss second and higher-order derivatives.

Second Derivatives

Because the derivative of a function is itself a function, we can take a derivative of the derivative. The derivative of a first derivative is called a **second derivative.** We can find the second derivative of a function f by differentiating it twice. If f' represents the first derivative of a function, then f'' represents the second derivative of that function.

● EXAMPLE 1 *Second Derivative*

(a) Find the second derivative of $y = x^4 - 3x^2 + x^{-2}$.
(b) If $f(x) = 3x^3 - 4x^2 + 5$, find $f''(x)$.

Solution

(a) The first derivative is $y' = 4x^3 - 6x - 2x^{-3}$.
The second derivative, which we may denote by y'', is

$$y'' = 12x^2 - 6 + 6x^{-4}$$

(b) The first derivative is $f'(x) = 9x^2 - 8x$.
The second derivative is $f''(x) = 18x - 8$.

It is also common to use $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^2f(x)}{dx^2}$ to denote the second derivative of a function.

● EXAMPLE 2 *Second Derivative*

If $y = \sqrt{2x - 1}$, find d^2y/dx^2.

Solution

The first derivative is

$$\frac{dy}{dx} = \frac{1}{2}(2x - 1)^{-1/2}(2) = (2x - 1)^{-1/2}$$

The second derivative is

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(2x - 1)^{-3/2}(2) = -(2x - 1)^{-3/2}$$

$$= \frac{-1}{(2x - 1)^{3/2}} = \frac{-1}{\sqrt{(2x - 1)^3}}$$

Higher-Order Derivatives

We can also find third, fourth, fifth, and higher derivatives, continuing indefinitely. The third, fourth, and fifth derivatives of a function f are denoted by f''', $f^{(4)}$, and $f^{(5)}$, respectively. Other notations for the third and fourth derivatives include

$$y''' = \frac{d^3y}{dx^3} = \frac{d^3f(x)}{dx^3}, \qquad y^{(4)} = \frac{d^4y}{dx^4} = \frac{d^4f(x)}{dx^4}$$

● EXAMPLE 3 *Higher-Order Derivatives*

Find the first four derivatives of $f(x) = 4x^3 + 5x^2 + 3$.

Solution

$$f'(x) = 12x^2 + 10x, \qquad f''(x) = 24x + 10, \qquad f'''(x) = 24, \qquad f^{(4)}(x) = 0$$

Just as the first derivative, $f'(x)$, can be used to determine the rate of change of a function $f(x)$, the second derivative, $f''(x)$, can be used to determine the rate of change of $f'(x)$.

EXAMPLE 4 *Worldwide Cellular Subscriberships (Application Preview)*

By using the International Telecommunications Union data, the millions of worldwide cellular subscribers (both actual and projected) can be modeled by

$$C(t) = -0.895t^3 + 30.6t^2 + 2.99t + 55.6$$

where t is the number of years past 1995.

(a) Find the instantaneous rate of change of the worldwide cellular subscribers function.
(b) Find the instantaneous rate of change of the function found in part (a).
(c) Find where the function in part (b) equals zero. Then explain how the rate of change of the number of cellular subscribers is changing for one t-value before and one after this value.

Solution

(a) The instantaneous rate of change of $C(t)$ is

$$C'(t) = -2.685t^2 + 61.2t + 2.99$$

(b) The instantaneous rate of change of $C'(t)$ is

$$C''(t) = -5.37t + 61.2$$

(c)

$$0 = -5.37t + 61.2$$
$$5.37t = 61.2$$
$$t = \frac{61.2}{5.37} \approx 11.4$$

For $t = 11$, which is less than 11.4,

$$C''(11) = -5.37(11) + 61.2 = 2.13$$

This means that for $t = 11$, the rate of change of the number of cellular subscribers is changing at the rate of 2.13 million subscribers per year, per year. Thus for $t = 11$ the rate of change of the number of cellular subscribers is increasing, so the number of cellular subscribers is increasing at an increasing rate. See Figure 9.33(a) earlier in this section. For $t = 12$,

$$C''(12) = -5.37(12) + 61.2 = -3.24$$

This means that the rate of change of the number of cellular subscribers is changing at the rate of -3.24 million subscribers per year, per year. Thus for $t = 12$ the rate of change of the number of cellular subscribers is decreasing, so the number of cellular subscribers is (still) increasing but at a decreasing (slower) rate. See Figure 9.33(a).

● EXAMPLE 5 *Rate of Change of a Derivative*

Let $f(x) = 3x^4 + 6x^3 - 3x^2 + 4$.

(a) How fast is $f(x)$ changing at (1, 10)?
(b) How fast is $f'(x)$ changing at (1, 10)?
(c) Is $f'(x)$ increasing or decreasing at (1, 10)?

Solution

(a) Because $f'(x) = 12x^3 + 18x^2 - 6x$, we have

$$f'(1) = 12 + 18 - 6 = 24$$

Thus the rate of change of $f(x)$ at (1, 10) is 24 (y units per x unit).

(b) Because $f''(x) = 36x^2 + 36x - 6$, we have

$$f''(1) = 66$$

Thus the rate of change of $f'(x)$ at (1, 10) is 66 (y units per x unit per x unit).

(c) Because $f''(1) = 66 > 0, f'(x)$ is increasing at (1, 10).

● EXAMPLE 6 *Acceleration*

Suppose that a particle travels according to the equation

$$s = 100t - 16t^2 + 200$$

where s is the distance in feet and t is the time in seconds. Then ds/dt is the velocity, and $d^2s/dt^2 = dv/dt$ is the acceleration of the particle. Find the acceleration.

Solution

The velocity is $v = ds/dt = 100 - 32t$ feet per second, and the acceleration is

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = -32 \text{ (ft/sec)/sec} = -32 \text{ ft/sec}^2$$

● *Checkpoint*

Suppose that the distance a particle travels is given by

$$s = 4x^3 - 12x^2 + 6$$

where s is in feet and x is in seconds.

1. Find the function that describes the velocity of this particle.
2. Find the function that describes the acceleration of this particle.
3. Is the acceleration always positive?
4. When does the *velocity* of this particle increase?

Calculator Note

We can use the numerical derivative feature of a graphing calculator to find the second derivative of a function at a point. ■

● EXAMPLE 7 *Second Derivative*

Find $f''(2)$ if $f(x) = \sqrt{x^3 - 1}$.

Solution

We need the derivative of the derivative function, evaluated at $x = 2$. Figure 9.34 shows how the numerical derivative feature of a graphing calculator can be used to obtain this result.

```
nDeriv(nDeriv(√(
X^3-1),X,X),X,2)
                .323969225
```

Figure 9.34

Thus Figure 9.34 shows that $f''(2) = 0.323969225 \approx 0.32397$. We can check this result by calculating $f''(x)$ with formulas.

$$f'(x) = \frac{1}{2}(x^3 - 1)^{-\frac{1}{2}}(3x^2)$$

$$f''(x) = \frac{1}{2}(x^3 - 1)^{-\frac{1}{2}}(6x) + (3x^2)\left[-\frac{1}{4}(x^3 - 1)^{-\frac{3}{2}}(3x^2)\right]$$

$$f''(2) = 0.3239695483 \approx 0.32397$$

Thus we see that the numerical derivative approximation is quite accurate.

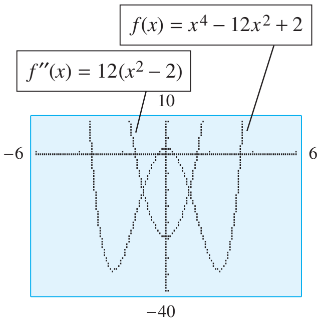

Figure 9.35

● EXAMPLE 8 *Second Derivatives and Graphs*

(a) Given $f(x) = x^4 - 12x^2 + 2$, graph $f(x)$ and its second derivative on the same set of axes over an interval that contains all x-values where $f''(x) = 0$.

(b) When the graph of $y = f(x)$ is opening downward, is $f''(x) > 0$, $f''(x) < 0$, or $f''(x) = 0$?

(c) When the graph of $y = f(x)$ is opening upward, is $f''(x) > 0$, $f''(x) < 0$, or $f''(x) = 0$?

Solution

(a) $f'(x) = 4x^3 - 24x$ and $f''(x) = 12x^2 - 24 = 12(x^2 - 2)$. Because $f''(x) = 0$ at $x = -\sqrt{2}$ and at $x = \sqrt{2}$, we use an x-range that contains $x = -\sqrt{2}$ and $\sqrt{2}$. The graph of $f(x) = x^4 - 12x^2 + 2$ and its second derivative, $f''(x) = 12x^2 - 24$, are shown in Figure 9.35.

(b) The graph of $y = f(x)$ appears to be opening downward on the same interval for which $f''(x) < 0$.

(c) The graph appears to be opening upward on the same intervals for which $f''(x) > 0$.

● *Checkpoint Solutions*

1. The velocity is described by $s'(x) = 12x^2 - 24x$.
2. The acceleration is described by $s''(x) = 24x - 24$.
3. No; the acceleration is positive when $s''(x) > 0$—that is, when $24x - 24 > 0$. It is zero when $24x - 24 = 0$ and negative when $24x - 24 < 0$. Thus acceleration is negative when $x < 1$ second, zero when $x = 1$ second, and positive when $x > 1$ second.
4. The velocity increases when the acceleration is positive. Thus the velocity is increasing after 1 second.

9.8 Exercises

In Problems 1–6, find the second derivative.

1. $f(x) = 2x^{10} - 18x^5 - 12x^3 + 4$
2. $y = 6x^5 - 3x^4 + 12x^2$
3. $g(x) = x^3 - \dfrac{1}{x}$
4. $h(x) = x^2 - \dfrac{1}{x^2}$
5. $y = x^3 - \sqrt{x}$
6. $y = 3x^2 - \sqrt[3]{x^2}$

In Problems 7–12, find the third derivative.

7. $y = x^5 - 16x^3 + 12$
8. $y = 6x^3 - 12x^2 + 6x$
9. $f(x) = 2x^9 - 6x^6$
10. $f(x) = 3x^5 - x^6$
11. $y = 1/x$
12. $y = 1/x^2$

In Problems 13–24, find the indicated derivative.

13. If $y = x^5 - x^{1/2}$, find $\dfrac{d^2y}{dx^2}$.
14. If $y = x^4 + x^{1/3}$, find $\dfrac{d^2y}{dx^2}$.
15. If $f(x) = \sqrt{x + 1}$, find $f'''(x)$.
16. If $f(x) = \sqrt{x - 5}$, find $f'''(x)$.
17. Find $\dfrac{d^4y}{dx^4}$ if $y = 4x^3 - 16x$.
18. Find $y^{(4)}$ if $y = x^6 - 15x^3$.

19. Find $f^{(4)}(x)$ if $f(x) = \sqrt{x}$.
20. Find $f^{(4)}(x)$ if $f(x) = 1/x$.
21. Find $y^{(4)}$ if $y' = \sqrt{4x - 1}$.
22. Find $y^{(5)}$ if $\dfrac{d^2y}{dx^2} = \sqrt[3]{3x + 2}$.
23. Find $f^{(6)}(x)$ if $f^{(4)}(x) = x(x + 1)^{-1}$.
24. Find $f^{(3)}(x)$ if $f'(x) = \dfrac{x^2}{x^2 + 1}$.
25. If $f(x) = 16x^2 - x^3$, what is the rate of change of $f'(x)$ at (1, 15)?
26. If $y = 36x^2 - 6x^3 + x$, what is the rate of change of y' at (1, 31)?

In Problems 27–30, use the numerical derivative feature of a graphing utility to approximate the given second derivatives.

27. $f''(3)$ for $f(x) = x^3 - \dfrac{27}{x}$
28. $f''(-1)$ for $f(x) = \dfrac{x^2}{4} - \dfrac{4}{x^2}$
29. $f''(21)$ for $f(x) = \sqrt{x^2 + 4}$
30. $f''(3)$ for $f(x) = \dfrac{1}{\sqrt{x^2 + 7}}$

In Problems 31–34, do the following for each function $f(x)$.
(a) Find $f'(x)$ and $f''(x)$.
(b) Graph $f(x)$, $f'(x)$, and $f''(x)$ with a graphing utility.
(c) Identify x-values where $f''(x) = 0$, $f''(x) > 0$, and $f''(x) < 0$.
(d) Identify x-values where $f'(x)$ has a maximum point or a minimum point, where $f'(x)$ is increasing, and where $f'(x)$ is decreasing.
(e) When $f(x)$ has a maximum point, is $f''(x) > 0$ or $f''(x) < 0$?
(f) When $f(x)$ has a minimum point, is $f''(x) > 0$ or $f''(x) < 0$?

31. $f(x) = x^3 - 3x^2 + 5$
32. $f(x) = 2 + 3x - x^3$
33. $f(x) = -\dfrac{1}{3}x^3 - x^2 + 3x + 7$
34. $f(x) = \dfrac{1}{3}x^3 - \dfrac{3x^2}{2} - 4x + 10$

APPLICATIONS

35. ***Acceleration*** A particle travels as a function of time according to the formula

$$s = 100 + 10t + 0.01t^3$$

where s is in meters and t is in seconds. Find the acceleration of the particle when $t = 2$.

36. ***Acceleration*** If the formula describing the distance s (in feet) an object travels as a function of time (in seconds) is

$$s = 100 + 160t - 16t^2$$

what is the acceleration of the object when $t = 4$?

37. ***Revenue*** The revenue (in dollars) from the sale of x units of a certain product can be described by

$$R(x) = 100x - 0.01x^2$$

Find the instantaneous rate of change of the marginal revenue.

38. ***Revenue*** Suppose that the revenue (in dollars) from the sale of a product is given by

$$R = 70x + 0.5x^2 - 0.001x^3$$

where x is the number of units sold. How fast is the marginal revenue $\overline{MR}$ changing when $x = 100$?

39. ***Sensitivity*** When medicine is administered, reaction (measured in change of blood pressure or temperature) can be modeled by

$$R = m^2\left(\frac{c}{2} - \frac{m}{3}\right)$$

where c is a positive constant and m is the amount of medicine absorbed into the blood (*Source:* R. M. Thrall et al., *Some Mathematical Models in Biology,* U.S. Department of Commerce, 1967). The sensitivity to the medication is defined to be the rate of change of reaction R with respect to the amount of medicine m absorbed in the blood.
(a) Find the sensitivity.
(b) Find the instantaneous rate of change of sensitivity with respect to the amount of medicine absorbed in the blood.
(c) Which order derivative of reaction gives the rate of change of sensitivity?

40. ***Photosynthesis*** The amount of photosynthesis that takes place in a certain plant depends on the intensity of light x according to the equation

$$f(x) = 145x^2 - 30x^3$$

(a) Find the rate of change of photosynthesis with respect to the intensity.
(b) What is the rate of change when $x = 1$? when $x = 3$?
(c) How fast is the rate found in part (a) changing when $x = 1$? when $x = 3$?