Chapter 3 Vectors (page 59)

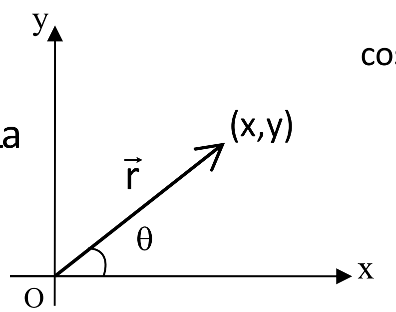
Problems: 4, 12, 21, 23, 37, 51

3.1 Coordinate Systems

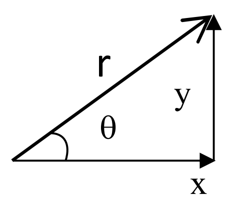
To describe object's location or **object's motion** requires describing **the object's position at various times.**

Cartesian coordinate system or ***rectangular coordinates***, in which horizontal and vertical axes intersect at a point defined as the origin (Fig.1). Any point (x,y) can be expressed in a plane by its ***polar coordinates*** **(*r*, θ)**, as shown in Figure 1a. In this *polar coordinate system*, ***r*** is **the distance from the origin to the point** having **Cartesian coordinates (*x*, *y*)**, and **θ is the angle** between a line drawn from the origin to the point and a **fixed axis x-axis**.

Fig.1a



$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$



$$x = r\cos\theta$$

$$y = r\sin\theta$$

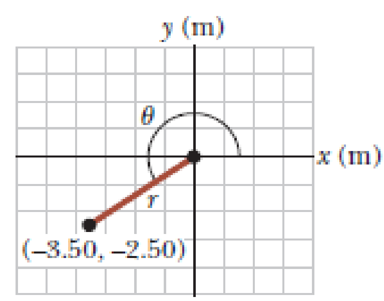
$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$, $\tan\theta = \dfrac{y}{x}$

Example 3.1 Polar Coordinates

The Cartesian coordinates of a point in the *xy* plane are $(x, y) =$ (- 3.50, -2.50) m as shown in Figure 3.3. Find the polar coordinates of this point.



Solution From the Equations

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.5)^2 + (-2.5)^2} = 4.30\text{m}$$

$$\tan\theta = \frac{y}{x} = \frac{-2.5}{-3.5} = 0.714$$

$$\theta = \tan^{-1}(0.714) = 35.5 + 180 = 215.5^\circ$$

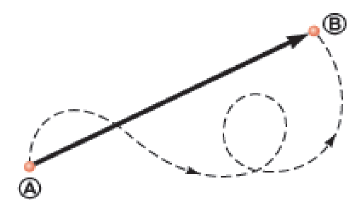
The signs of *x* and *y* determine where the point lies. It lies in the third quadrant of the coordinate system. (x ,y) (-,-)

3.2 Vector and Scalar Quantities

Physical quantities are **scalar quantities** or **vector quantities**

A scalar quantity is completely specified by a single value with an appropriate unit and has no direction. (has only magnitude)
Examples of scalar quantities are volume, mass, speed, and time

A vector quantity is completely specified by a number and appropriate units plus a direction.

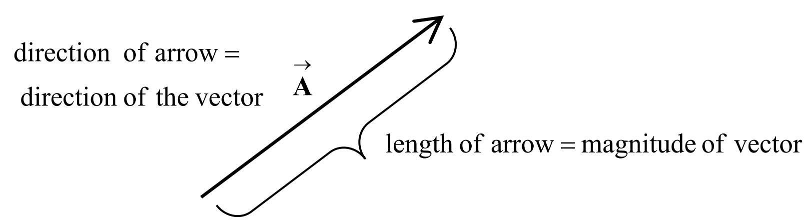


Vector quantities: displacement; **Displacement depends only on the initial and final positions, so the displacement vector is independent of the path taken between these two points.**

Symbol: **boldface letter A** (A vector).

Arrow is written over the symbol, vector: $\vec{A}$.

The magnitude of the vector A is written either *A* or |**A**|.



3.3 Some Properties of Vectors

Equality of Two Vectors

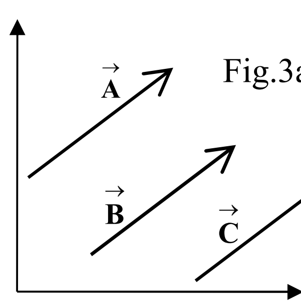
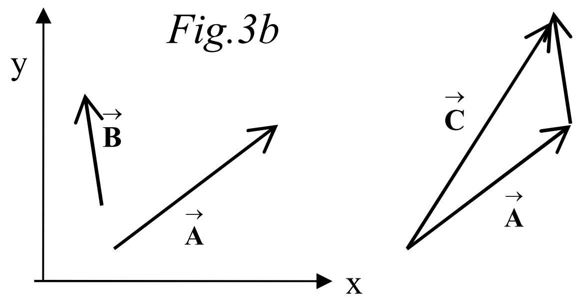


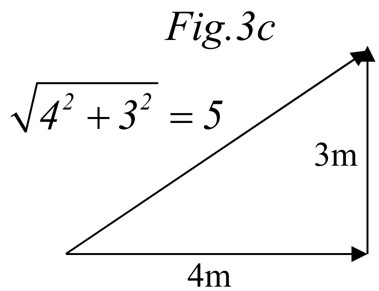
Fig.3a

Two vectors **A** and **B** may be **defined to be equal** if they have the **same magnitude and point in the same direction.** That is, **A** = **B** only if *A* = *B* and if **A** and **B** point in the same direction along parallel lines.

Adding Vectors

Graphical methods: To add vector **B** to vector **A**, first draw vector **A** on graph paper, with its magnitude represented by a convenient length scale, and then draw vector **B** to the same scale with its tail starting from the tip of **A**, as shown in **Figure 3b**. The **resultant vector R** = **A** + **B** is the vector drawn from the **tail of A to the tip of B**. **Example** of adding two displacements: **4 m east, 3 m north** shown in Fig.3c.





For the case of four vectors, the resultant vector

R = **A** + **B** + **C** + **D,**

R is the vector drawn from the tail of the first vector to the tip of the last vector, and completes the polygon Fig.4 .

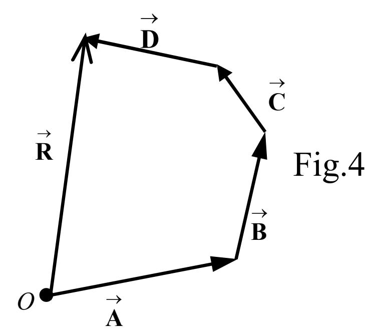
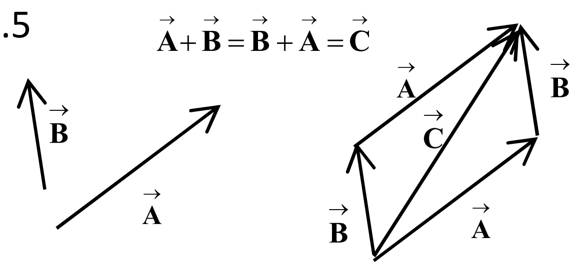


Fig.5



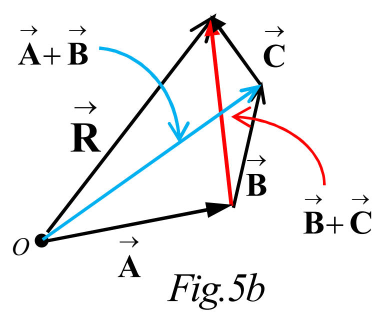
Vector sum is independent of the order of the addition.

Commutative law of addition **Fig. 5a**:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Associative law of addition: (Fig. 5b)

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$



Negative of a Vector

The negative of the vector **A** is defined as the vector that when added to A gives zero for the vector sum. That is,

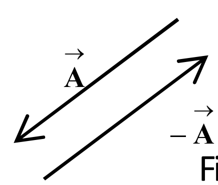


Fig.6 a

A + (- **A**) = 0.

The vectors **A** and **– A** have the same magnitude but point in opposite directions.

Subtracting Vectors

We define the operation **A** - **B** as vector - **B** added to vector **A**:

A - **B** = **A** + (- **B**)

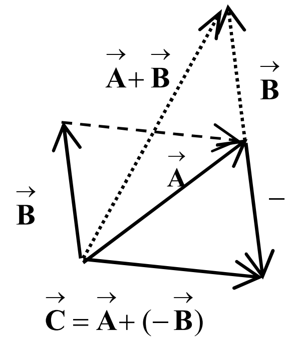
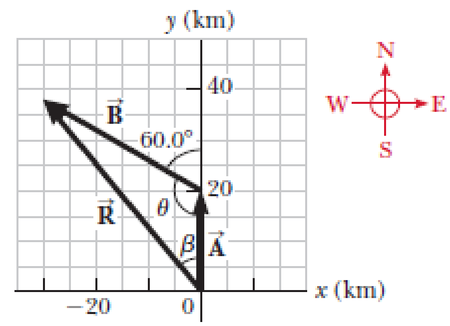


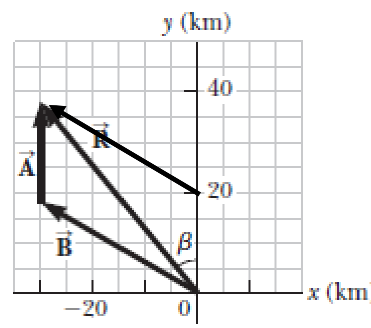
Fig.6b

The geometric construction for subtracting two vectors in this way is illustrated in the Figure 6b.

Example 3.2 A Vacation Trip

A car travels 20.0 km due **north** and then 35.0 km in a direction **60.0° west of north** as shown in Figure 3.11a. Find the magnitude and direction of the car's resultant displacement.





Solution

Geometrically, using graph paper and with an ordinary ruler and protractor, measure the magnitude of R and its direction algebraically. The magnitude of **R** can be obtained from the law of cosines as applied to the triangle.

With θ=180° - 60° = 120°

$$R=\sqrt{A^2+B^2+2AB\cos\theta}\,,\quad R=\sqrt{A^2+B^2+2AB\cos 60}$$

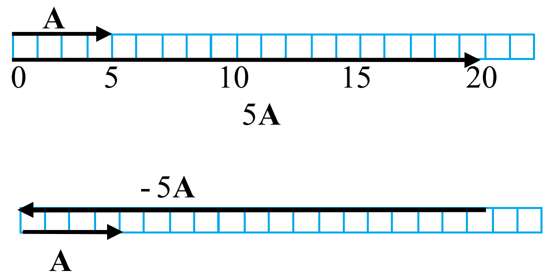
$$=\sqrt{20^2+35^2-2\times20\times35\cos120^\circ}=48.2\text{km}\qquad \textbf{[with – put cos120]}$$

Same to put $R=\sqrt{A^2+B^2+2AB\cos 60}$ **[with + put cos60]**

$$\frac{\sin\beta}{B}=\frac{\sin\theta}{R}\quad\Rightarrow\sin\beta=B\times\frac{\sin\theta}{R}=35\times\frac{\sin120^\circ}{48.2}=0.629\,,\qquad \beta=39^\circ$$

Adding the **vectors in reverse order** (**B** + **A**) gives the **same result** for **R**. (Fig. b)

Multiplying a Vector by a Scalar



If m is a positive scalar quantity, then the product <u>*m***A**</u> is a vector that has the same direction as **A** and <u>magnitude *mA*</u>.
If vector **A** is multiplied by **a negative** scalar quantity <u>*–m*</u>, then the product – *m***A**, is directed <u>opposite **A**</u>.

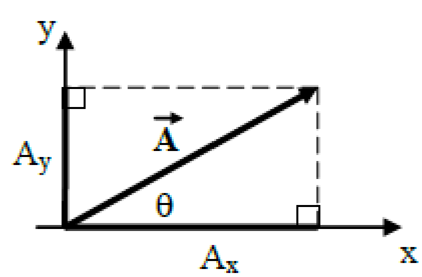
3.4 Components of a Vector and Unit Vectors

Adding vectors can be done using the components **of** the vectors.

A vector **A** can be expressed as the sum of two other vectors $\mathbf{A}_x$ and $\mathbf{A}_y$. The three vectors form a right triangle such that $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$.

The "components of a vector **A**, are " written A_x and A_y (without the boldface notation). These components represent projection of A along the *corresponding axis*.

These components form two sides of a right triangle with a hypotenuse of length *A*. so



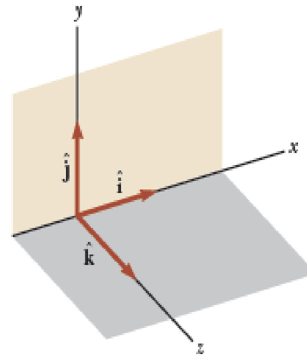
$$\cos\theta = \frac{A_x}{A} \Rightarrow \qquad A_x = A\cos\theta$$

$$\sin\theta = \frac{A_y}{A} \Rightarrow \qquad A_y = A\sin\theta$$

The magnitude and direction of **A** in terms of its components is given by the expressions

$$A = \sqrt{A_x^{\ 2} + A_y^{\ 2}}, \qquad \tan\theta = \frac{A_y}{A_x}, \qquad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

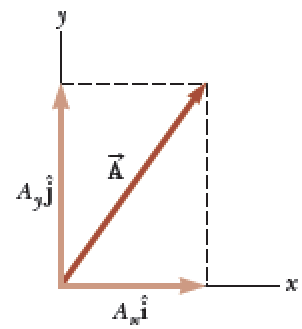
Note that the signs of the components of A determine in which quadrant the vector lies and which angle should take: **For example if** $\tan\theta = 1.73$, θ = 60º, 240º, if both the components is negative θ must equal 240º.



Unit Vectors

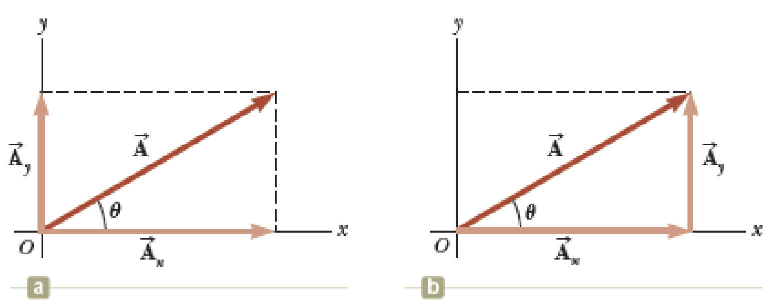
Unit-vectors: **a unit vector points in a certain direction but has a magnitude of 1**.

The symbols $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are used to represent unit vectors pointing in the positive *x*, *y*, and *z* directions, respectively.



Consider a vector **A** lying in the *xy* plane, as shown in Figure.

This vector can be expressed as the sum of two other vectors $\mathbf{A}_x$ and $\mathbf{A}_y$ such that $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$.



The components of **A** are scalars written as A_x and A_y.

The vector $\mathbf{A}_x$ can be represented as the product of the component A_x and the unit vector $\hat{\mathbf{i}}$.

The vector $\mathbf{A}_y$ can be represented as the product of the component A_x and the unit vector $\hat{\mathbf{j}}$.

In the unit–vector notation, the vector $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$ is given by

$$\mathbf{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}$$

The position of a point in the *xy* plane with Cartesian coordinates (*x*, *y*), can be specified by the position vector **r**, which in unit–vector form is given by

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

To add vector **B** to vector **A**, we add the *x* and *y* components separately. The resultant vector **R** = **A** + **B** is therefore

$$\mathbf{R} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}) + (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}})$$

$$\mathbf{R} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}}$$ **but** $\mathbf{R} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$; therefore

$R_x = A_x + B_x$ **and** $R_y = A_y + B_y$

The magnitude of R and the angle it makes with the *x* axis is obtained from

$$R = \sqrt{R_x^{\ 2} + R_y^{\ 2}},$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x}$$

For 2 vectors **A**, **B** in three dimensions

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \qquad \mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\mathbf{R} = (A_x\hat{\mathbf{i}} + B_x\hat{\mathbf{i}}) + (A_y\hat{\mathbf{j}} + B_y\hat{\mathbf{j}}) + (A_z\hat{\mathbf{k}} + B_z\hat{\mathbf{k}})$$

The angle θ_x that **R** makes with the *x* axis

$$\cos\theta_x = \frac{R_x}{R}$$

And magnitude: $R = \sqrt{R_x^{\ 2} + R_y^{\ 2} + R_z^{\ 2}}$

Example 3.3 The Sum of Two Vectors

Find the sum of two vectors A and B lying in the *xy* plane and given by

$\mathbf{A} = (2.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})\,\text{m}$ $\qquad\qquad$ $\mathbf{B} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}})\,\text{m}$

Solution $\quad \mathbf{R} = (2.0 + 2.0)\hat{\mathbf{i}} + (2.0 - 4.0)\hat{\mathbf{j}} = 4.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}}\ \text{m}$

$R_x = 4.0\text{m}$ **and** $R_y = -2.0\text{m}$

$$R = \sqrt{R_x^{\,2} + R_y^{\,2}} = \sqrt{4^2 + (-2)^2} = \sqrt{20} = 4.5\,\text{m}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2}{4} = -0.5 \quad , \quad \theta = -27^\circ \text{ or } \theta = 333^\circ$$

(This is correct if θ is measured clockwise relative to the x axis, but with the standard convention, it is to be measured counterclockwise from +x axis $\theta = 333^\circ$)

Example 3.4 The Resultant Displacement

A particle undergoes three consecutive displacements:

$\Delta\mathbf{r}_1 = (15\hat{\mathbf{i}} + 30\hat{\mathbf{j}} + 12\hat{\mathbf{k}})\ \text{cm}$,

$\Delta\mathbf{r}_2 = (23\hat{\mathbf{i}} - 14\hat{\mathbf{j}} - 5.0\hat{\mathbf{k}})\ \text{cm}$, and

$\Delta\mathbf{r}_3 = (-13\hat{\mathbf{i}} + 15\hat{\mathbf{j}})\text{cm}$.

Find the components of the **resultant displacement** and **its magnitude.**

Solution

the final position of the particle **from the origin** defines **the resultant displacement R .** To find **the resultant displacement**, **add the three vectors: [R is the vector from start to end]**

$$\mathbf{R} = \Delta\mathbf{r}_1 + \Delta\mathbf{r}_2 + \Delta\mathbf{r}_3$$

$$= (15 + 23 - 13)\hat{\mathbf{i}} + (30 - 14 + 15)\hat{\mathbf{j}} + (12 - 5.0)\hat{\mathbf{k}}\ \text{cm}$$

$= (25\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 7.0\hat{\mathbf{k}})\ \text{cm}$, $R_x = 25\text{cm}$, $R_y = 31\text{cm}$ **and** $R_z = 7.0\text{cm}$

$$R=\sqrt{R_x^{\,2}+R_y^{\,2}+R_z^{\,2}}=\sqrt{25^2+31^2+7.0^2}=40\text{cm}$$

$$\Delta\mathbf{r}_1=15\hat{i}+30\hat{j}+12\hat{k},$$
$$\Delta\mathbf{r}_2=23\hat{i}-14\hat{j}-5\hat{k}$$
$$\Delta\mathbf{r}_3=-13\hat{i}+15\hat{j} \qquad \text{all are in cm}$$

$$\mathbf{R}=(15+23-13)\hat{i}+(30-14+15)\hat{j}+(12-5)\hat{k}$$

Example 3.5 Taking a Hike

A hiker begins a trip by first walking **25.0 km southeast** from her car. She stops and sets up her tent (خيمة) for the night. On the second day, **she walks 40.0 km** in a direction **60.0° north of east**, at which point she discovers a forest ranger's tower.

(A) Determine the components of the hiker's displacement for each day.

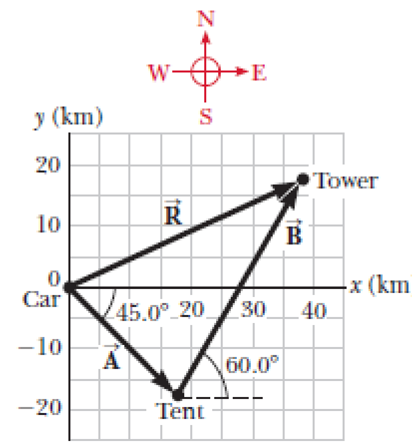


Figure 3.17 (Example 3.5) The total displacement of the hiker is the vector $\vec{R}=\vec{A}+\vec{B}$.

Solution

(A) Denote the displacement vectors on the first and second days by **A** and **B**, the resultant will be **R** as shown in the adjacent figure.

A has a <u>magnitude of 25.0 km and is directed 45.0° below</u> the positive *x* axis. **B** has a <u>magnitude of 40.0 km and is 60.0° north of east</u>. Adding A to **B** to get **R**:

$$A_x=A\cos\theta=25\cos(-45)=25(0.707)=17.7\text{km}$$

$$A_y=A\sin\theta=25\sin(-45)=25(-0.707)=-17.7\text{km}$$

$$B_x=B\cos\theta=40\cos(60)=40\,(0.5)=20\text{ km}$$

$$B_y=B\sin\theta=40\sin(60)=40\,(0.866)=34.6\text{km}$$

(B) Determine **the components of the hiker's resultant displacement R**

for the trip. Find an expression for **R** in terms of unit vectors.

$$R_x = A_x + B_x = 17.7 + 20 = 37.7\text{km}$$
$$R_y = A_y + B_y = -17.7 + 34.6 = 16.9\text{ km}$$
$$\mathbf{R} = (37.7\hat{\mathbf{i}} + 16.9\hat{\mathbf{j}})\text{km}$$

Problem: 3-51 A person going for a walk follows the path shown in Fig. P3.51. the total trip consists of four straight-line paths. At the end of the walk, what is the person's resultant displacement measured from the starting point.

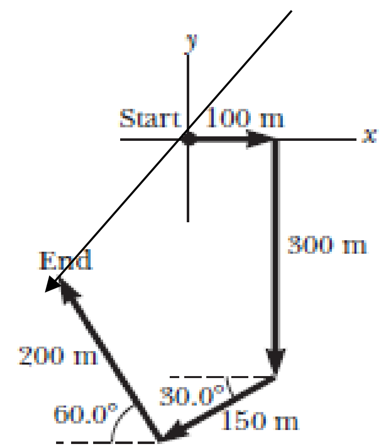


Figure P3.51

$$\vec{d}_1 = 100\hat{i}\ \text{ m},$$
$$\vec{d}_2 = (-300\hat{j})\text{m}$$
$$\vec{d}_3 = (-150\cos 30^\circ\,\hat{i} - 150\sin 30^\circ\,\hat{j}) \quad = (-130\,\hat{i} - 75\,\hat{j})\text{m}$$
$$\vec{d}_4 = (-200\ \cos 60^\circ\,\hat{i} + 200\ \sin 60^\circ\,\hat{j}) \ = (-100\ \hat{i} + 173\,\hat{j})\text{m}$$
$$\vec{d} = \ \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4 = (-130\,\hat{i} - 202\,\hat{j})\ \text{m}$$
$$d = \sqrt{(-130)^2 + (-202)^2} = 240\text{ m},$$
$$\theta = \tan^{-1}\left(\frac{-202}{-130}\right) = 57.2^\circ + 180 = 237.2^\circ$$

لاحظ الزاوية تقع في الربع الثالث وليس الاول لان الإزاحة السينية والصادية سالبتين