c. $3x^2 - e^x = 0$, where g is the function in Exercise 12(c) of Section 2.2.

d. $x - \cos x = 0$, where g is the function in Exercise 12(d) of Section 2.2.

13. The following sequences converge to 0. Use Aitken's Δ^2 method to generate $\{\hat{p}_n\}$ until $|\hat{p}_n| \le 5\times 10^{-2}$:

a. $p_n = \dfrac{1}{n}, \quad n \ge 1$ **b.** $p_n = \dfrac{1}{n^2}, \quad n \ge 1$

14. A sequence $\{p_n\}$ is said to be **superlinearly convergent** to p if

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0.$$

a. Show that if $p_n \to p$ of order α for $\alpha > 1$, then $\{p_n\}$ is superlinearly convergent to p.

b. Show that $p_n = \frac{1}{n^n}$ is superlinearly convergent to 0 but does not converge to 0 of order α for any $\alpha > 1$.

15. Suppose that $\{p_n\}$ is superlinearly convergent to p. Show that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = 1.$$

16. Prove Theorem 2.14. [*Hint:* Let $\delta_n = (p_{n+1} - p)/(p_n - p) - \lambda$, and show that $\lim_{n\to\infty} \delta_n = 0$. Then express $(\hat{p}_{n+1} - p)/(p_n - p)$ in terms of δ_n, δ_{n+1}, and λ.]

17. Let $P_n(x)$ be the nth Taylor polynomial for $f(x) = e^x$ expanded about $x_0 = 0$.

a. For fixed x, show that $p_n = P_n(x)$ satisfies the hypotheses of Theorem 2.14.

b. Let $x = 1$, and use Aitken's Δ^2 method to generate the sequence $\hat{p}_0, \dots, \hat{p}_8$.

c. Does Aitken's method accelerate convergence in this situation?

2.6 Zeros of Polynomials and Müller's Method

A *polynomial of degree n* has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where the a_i's, called the *coefficients* of P, are constants and $a_n \ne 0$. The zero function, $P(x) = 0$ for all values of x, is considered a polynomial but is assigned no degree.

Algebraic Polynomials

Theorem 2.16 **(Fundamental Theorem of Algebra)**

If $P(x)$ is a polynomial of degree $n \ge 1$ with real or complex coefficients, then $P(x) = 0$ has at least one (possibly complex) root. ■

Although the Fundamental Theorem of Algebra is basic to any study of elementary functions, the usual proof requires techniques from the study of complex function theory. The reader is referred to [SaS], p. 155, for the culmination of a systematic development of the topics needed to prove the Theorem.

Example 1 Determine all the zeros of the polynomial $P(x) = x^3 - 5x^2 + 17x - 13$.

Solution It is easily verified that $P(1) = 1 - 5 + 17 - 13 = 0$. so $x = 1$ is a zero of P and $(x - 1)$ is a factor of the polynomial. Dividing $P(x)$ by $x - 1$ gives

$$P(x) = (x - 1)(x^2 - 4x + 13).$$

Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all time, proved the Fundamental Theorem of Algebra in his doctoral dissertation and published it in 1799. He published different proofs of this result throughout his lifetime, in 1815, 1816, and as late as 1848. The result had been stated, without proof, by Albert Girard (1595–1632), and partial proofs had been given by Jean d'Alembert (1717–1783), Euler, and Lagrange.

To determine the zeros of $x^2 - 4x + 13$ we use the quadratic formula in its standard form, which gives the complex zeros

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i.$$

Hence the third-degree polynomial $P(x)$ has three zeros, $x_1 = 1$, $x_2 = 2 - 3i$, and $x_2 = 2 + 3i$. ∎

In the preceding example we found that the third-degree polynomial had three distinct zeros. An important consequence of the Fundamental Theorem of Algebra is the following corollary. It states that this is always the case, provided that when the zeros are not distinct we count the number of zeros according to their multiplicities.

Corollary 2.17 If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then there exist unique constants $x_1, x_2, \ldots, x_k$, possibly complex, and unique positive integers $m_1, m_2, \ldots, m_k$, such that $\sum_{i=1}^{k} m_i = n$ and

$$P(x) = a_n(x - x_1)^{m_1}(x - x_2)^{m_2} \cdots (x - x_k)^{m_k}.$$ ∎

By Corollary 2.17 the collection of zeros of a polynomial is unique and, if each zero x_i is counted as many times as its multiplicity m_i, a polynomial of degree n has exactly n zeros.

The following corollary of the Fundamental Theorem of Algebra is used often in this section and in later chapters.

Corollary 2.18 Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If $x_1, x_2, \ldots, x_k$, with $k > n$, are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, 2, \ldots, k$, then $P(x) = Q(x)$ for all values of x. ∎

This result implies that to show that two polynomials of degree less than or equal to n are the same, we only need to show that they agree at $n + 1$ values. This will be frequently used, particularly in Chapters 3 and 8.

Horner's Method

William Horner (1786–1837) was a child prodigy who became headmaster of a school in Bristol at age 18. Horner's method for solving algebraic equations was published in 1819 in the Philosophical Transactions of the Royal Society.

To use Newton's method to locate approximate zeros of a polynomial $P(x)$, we need to evaluate $P(x)$ and $P'(x)$ at specified values. Since $P(x)$ and $P'(x)$ are both polynomials, computational efficiency requires that the evaluation of these functions be done in the nested manner discussed in Section 1.2. Horner's method incorporates this nesting technique, and, as a consequence, requires only n multiplications and n additions to evaluate an arbitrary nth-degree polynomial.

Theorem 2.19 **(Horner's Method)**

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Define $b_n = a_n$ and

$$b_k = a_k + b_{k+1}x_0, \quad \text{for } k = n-1, n-2, \ldots, 1, 0.$$

Then $b_0 = P(x_0)$. Moreover, if

$$Q(x) = b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1,$$

then

$$P(x) = (x - x_0)Q(x) + b_0. \qquad \blacksquare$$

Paolo Ruffini (1765–1822) had described a similar method which won him the gold medal from the Italian Mathematical Society for Science. Neither Ruffini nor Horner was the first to discover this method; it was known in China at least 500 years earlier.

Proof By the definition of $Q(x)$,

$$\begin{aligned}(x - x_0)Q(x) + b_0 &= (x - x_0)(b_n x^{n-1} + \cdots + b_2 x + b_1) + b_0\\ &= (b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2 x^2 + b_1 x)\\ &\quad - (b_n x_0 x^{n-1} + \cdots + b_2 x_0 x + b_1 x_0) + b_0\\ &= b_n x^n + (b_{n-1} - b_n x_0)x^{n-1} + \cdots + (b_1 - b_2 x_0)x + (b_0 - b_1 x_0).\end{aligned}$$

By the hypothesis, $b_n = a_n$ and $b_k - b_{k+1}x_0 = a_k$, so

$$(x - x_0)Q(x) + b_0 = P(x) \quad \text{and} \quad b_0 = P(x_0). \qquad \blacksquare\ \blacksquare\ \blacksquare$$

Example 2 Use Horner's method to evaluate $P(x) = 2x^4 - 3x^2 + 3x - 4$ at $x_0 = -2$.

Solution When we use hand calculation in Horner's method, we first construct a table, which suggests the *synthetic division* name that is often applied to the technique. For this problem, the table appears as follows:

	Coefficient of x^4	Coefficient of x^3	Coefficient of x^2	Coefficient of x	Constant term
$x_0 = -2$	$a_4 = 2$	$a_3 = 0$	$a_2 = -3$	$a_1 = 3$	$a_0 = -4$
		$b_4 x_0 = -4$	$b_3 x_0 = 8$	$b_2 x_0 = -10$	$b_1 x_0 = 14$
	$b_4 = 2$	$b_3 = -4$	$b_2 = 5$	$b_1 = -7$	$b_0 = 10$

So,

$$P(x) = (x + 2)(2x^3 - 4x^2 + 5x - 7) + 10. \qquad \blacksquare$$

The word synthetic has its roots in various languages. In standard English it generally provides the sense of something that is "false" or "substituted". But in mathematics it takes the form of something that is "grouped together". Synthetic geometry treats shapes as whole, rather than as individual objects, which is the style in analytic geometry. In synthetic division of polynomials, the various powers of the variables are not explicitly given but kept grouped together.

An additional advantage of using the Horner (or synthetic-division) procedure is that, since

$$P(x) = (x - x_0)Q(x) + b_0,$$

where

$$Q(x) = b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1,$$

differentiating with respect to x gives

$$P'(x) = Q(x) + (x - x_0)Q'(x) \quad \text{and} \quad P'(x_0) = Q(x_0). \tag{2.16}$$

When the Newton-Raphson method is being used to find an approximate zero of a polynomial, $P(x)$ and $P'(x)$ can be evaluated in the same manner.

Example 3 Find an approximation to a zero of

$$P(x) = 2x^4 - 3x^2 + 3x - 4,$$

using Newton's method with $x_0 = -2$ and synthetic division to evaluate $P(x_n)$ and $P'(x_n)$ for each iterate x_n.

Solution With $x_0 = -2$ as an initial approximation, we obtained $P(-2)$ in Example 1 by

$$\begin{array}{r|rrrrrl} x_0 = -2 & 2 & 0 & -3 & 3 & -4 & \\ & & -4 & 8 & -10 & 14 & \\ \hline & 2 & -4 & 5 & -7 & 10 & = P(-2). \end{array}$$

Using Theorem 2.19 and Eq. (2.16),

$$Q(x) = 2x^3 - 4x^2 + 5x - 7 \quad \text{and} \quad P'(-2) = Q(-2),$$

so $P'(-2)$ can be found by evaluating $Q(-2)$ in a similar manner:

$$\begin{array}{r|rrrrl} x_0 = -2 & 2 & -4 & 5 & -7 & \\ & & -4 & 16 & -42 & \\ \hline & 2 & -8 & 21 & -49 & = Q(-2) = P'(-2) \end{array}$$

and

$$x_1 = x_0 - \frac{P(x_0)}{P'(x_0)} = x_0 - \frac{P(x_0)}{Q(x_0)} = -2 - \frac{10}{-49} \approx -1.796.$$

Repeating the procedure to find x_2 gives

$$\begin{array}{r|rrrrrl} -1.796 & 2 & 0 & -3 & 3 & -4 & \\ & & -3.592 & 6.451 & -6.197 & 5.742 & \\ \hline & 2 & -3.592 & 3.451 & -3.197 & 1.742 & = P(x_1) \\ & & -3.592 & 12.902 & -29.368 & & \\ \hline & 2 & -7.184 & 16.353 & -32.565 & = Q(x_1) & = P'(x_1). \end{array}$$

So $P(-1.796) = 1.742$, $P'(-1.796) = Q(-1.796) = -32.565$, and

$$x_2 = -1.796 - \frac{1.742}{-32.565} \approx -1.7425.$$

In a similar manner, $x_3 = -1.73897$, and an actual zero to five decimal places is -1.73896.

Note that the polynomial $Q(x)$ depends on the approximation being used and changes from iterate to iterate. ■

Algorithm 2.7 computes $P(x_0)$ and $P'(x_0)$ using Horner's method.

ALGORITHM 2.7

Horner's

To evaluate the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = (x - x_0)Q(x) + b_0$$

and its derivative at x_0:

INPUT degree n; coefficients $a_0, a_1, \ldots, a_n$; x_0.

OUTPUT $y = P(x_0)$; $z = P'(x_0)$.

Step 1 Set $y = a_n$; (*Compute b_n for P.*)
$z = a_n$. (*Compute b_{n-1} for Q.*)

Step 2 For $j = n-1, n-2, \ldots, 1$
set $y = x_0 y + a_j$; (*Compute b_j for P.*)
$z = x_0 z + y$. (*Compute b_{j-1} for Q.*)

Step 3 Set $y = x_0 y + a_0$. (*Compute b_0 for P.*)

Step 4 OUTPUT (y, z);
STOP. ■

If the Nth iterate, x_N, in Newton's method is an approximate zero for P, then

$$P(x) = (x - x_N)Q(x) + b_0 = (x - x_N)Q(x) + P(x_N) \approx (x - x_N)Q(x),$$

so $x - x_N$ is an approximate factor of $P(x)$. Letting $\hat{x}_1 = x_N$ be the approximate zero of P and $Q_1(x) \equiv Q(x)$ be the approximate factor gives

$$P(x) \approx (x - \hat{x}_1)Q_1(x).$$

We can find a second approximate zero of P by applying Newton's method to $Q_1(x)$.

If $P(x)$ is an nth-degree polynomial with n real zeros, this procedure applied repeatedly will eventually result in $(n-2)$ approximate zeros of P and an approximate quadratic factor $Q_{n-2}(x)$. At this stage, $Q_{n-2}(x) = 0$ can be solved by the quadratic formula to find the last two approximate zeros of P. Although this method can be used to find all the approximate zeros, it depends on repeated use of approximations and can lead to inaccurate results.

The procedure just described is called **deflation**. The accuracy difficulty with deflation is due to the fact that, when we obtain the approximate zeros of $P(x)$, Newton's method is used on the reduced polynomial $Q_k(x)$, that is, the polynomial having the property that

$$P(x) \approx (x - \hat{x}_1)(x - \hat{x}_2) \cdots (x - \hat{x}_k)Q_k(x).$$

An approximate zero $\hat{x}_{k+1}$ of Q_k will generally not approximate a root of $P(x) = 0$ as well as it does a root of the reduced equation $Q_k(x) = 0$, and inaccuracy increases as k increases. One way to eliminate this difficulty is to use the reduced equations to find approximations $\hat{x}_2$, $\hat{x}_3, \ldots, \hat{x}_k$ to the zeros of P, and then improve these approximations by applying Newton's method to the original polynomial $P(x)$.

Complex Zeros: Müller's Method

One problem with applying the Secant, False Position, or Newton's method to polynomials is the possibility of the polynomial having complex roots even when all the coefficients are real numbers. If the initial approximation is a real number, all subsequent approximations will also be real numbers. One way to overcome this difficulty is to begin with a complex initial approximation and do all the computations using complex arithmetic. An alternative approach has its basis in the following theorem.

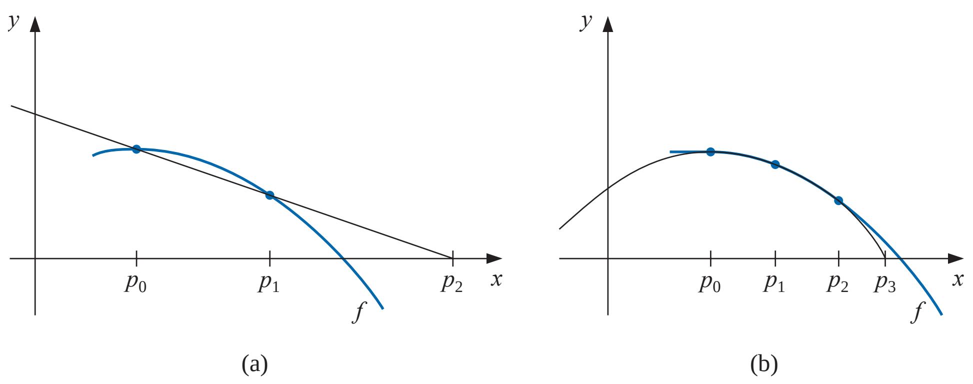
Theorem 2.20 If $z = a+bi$ is a complex zero of multiplicity m of the polynomial $P(x)$ with real coefficients, then $\bar{z} = a - bi$ is also a zero of multiplicity m of the polynomial $P(x)$, and $(x^2 - 2ax + a^2 + b^2)^m$ is a factor of $P(x)$. ■

Müller's method is similar to the Secant method. But whereas the Secant method uses a line through two points on the curve to approximate the root, Müller's method uses a parabola through three points on the curve for the approximation.

A synthetic division involving quadratic polynomials can be devised to approximately factor the polynomial so that one term will be a quadratic polynomial whose complex roots are approximations to the roots of the original polynomial. This technique was described in some detail in our second edition [BFR]. Instead of proceeding along these lines, we will now consider a method first presented by D. E. Müller [Mu]. This technique can be used for any root-finding problem, but it is particularly useful for approximating the roots of polynomials.

The Secant method begins with two initial approximations p_0 and p_1 and determines the next approximation p_2 as the intersection of the x-axis with the line through $(p_0, f(p_0))$ and $(p_1, f(p_1))$. (See Figure 2.13(a).) Müller's method uses three initial approximations, p_0, p_1, and p_2, and determines the next approximation p_3 by considering the intersection of the x-axis with the parabola through $(p_0, f(p_0))$, $(p_1, f(p_1))$, and $(p_2, f(p_2))$. (See Figure 2.13(b).)

Figure 2.13



The derivation of Müller's method begins by considering the quadratic polynomial

$$P(x) = a(x - p_2)^2 + b(x - p_2) + c$$

that passes through $(p_0, f(p_0))$, $(p_1, f(p_1))$, and $(p_2, f(p_2))$. The constants a, b, and c can be determined from the conditions

$$f(p_0) = a(p_0 - p_2)^2 + b(p_0 - p_2) + c, \qquad (2.17)$$

$$f(p_1) = a(p_1 - p_2)^2 + b(p_1 - p_2) + c, \qquad (2.18)$$

and

$$f(p_2) = a \cdot 0^2 + b \cdot 0 + c = c \qquad (2.19)$$

to be

$$c = f(p_2), \tag{2.20}$$

$$b = \frac{(p_0 - p_2)^2[f(p_1) - f(p_2)] - (p_1 - p_2)^2[f(p_0) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)}, \tag{2.21}$$

and

$$a = \frac{(p_1 - p_2)[f(p_0) - f(p_2)] - (p_0 - p_2)[f(p_1) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)}. \tag{2.22}$$

To determine p_3, a zero of P, we apply the quadratic formula to $P(x) = 0$. However, because of round-off error problems caused by the subtraction of nearly equal numbers, we apply the formula in the manner prescribed in Eq (1.2) and (1.3) of Section 1.2:

$$p_3 - p_2 = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}.$$

This formula gives two possibilities for p_3, depending on the sign preceding the radical term. In Müller's method, the sign is chosen to agree with the sign of b. Chosen in this manner, the denominator will be the largest in magnitude and will result in p_3 being selected as the closest zero of P to p_2. Thus

$$p_3 = p_2 - \frac{2c}{b + \operatorname{sgn}(b)\sqrt{b^2 - 4ac}},$$

where a, b, and c are given in Eqs. (2.20) through (2.22).

Once p_3 is determined, the procedure is reinitialized using p_1, p_2, and p_3 in place of p_0, p_1, and p_2 to determine the next approximation, p_4. The method continues until a satisfactory conclusion is obtained. At each step, the method involves the radical $\sqrt{b^2 - 4ac}$, so the method gives approximate complex roots when $b^2 - 4ac < 0$. Algorithm 2.8 implements this procedure.

ALGORITHM
2.8

Müller's

To find a solution to $f(x) = 0$ given three approximations, p_0, p_1, and p_2:

INPUT p_0, p_1, p_2; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $h_1 = p_1 - p_0$;
$h_2 = p_2 - p_1$;
$\delta_1 = (f(p_1) - f(p_0))/h_1$;
$\delta_2 = (f(p_2) - f(p_1))/h_2$;
$d = (\delta_2 - \delta_1)/(h_2 + h_1)$;
$i = 3$.

Step 2 While $i \le N_0$ do Steps 3–7.

Step 3 $b = \delta_2 + h_2 d$;
$D = (b^2 - 4f(p_2)d)^{1/2}$. (*Note: May require complex arithmetic.*)

Step 4 If $|b - D| < |b + D|$ then set $E = b + D$
else set $E = b - D$.

Step 5 Set $h = -2f(p_2)/E$;
$p = p_2 + h$.

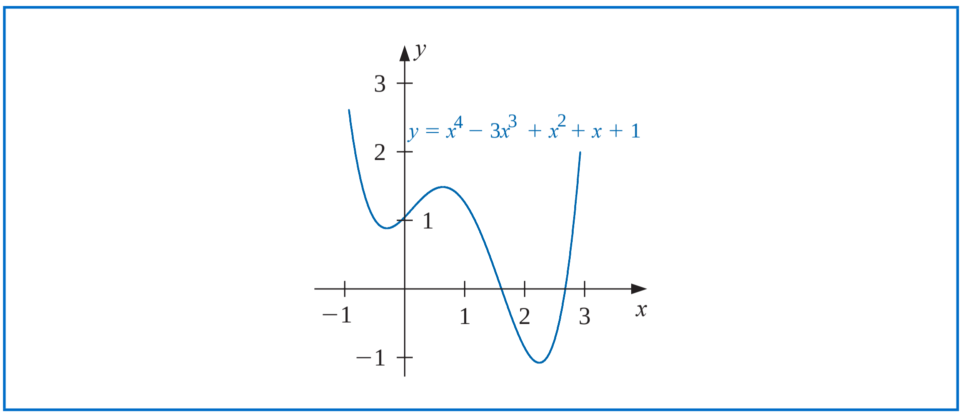
Step 6 If $|h| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 7 Set $p_0 = p_1$; (*Prepare for next iteration.*)
$p_1 = p_2$;
$p_2 = p$;
$h_1 = p_1 - p_0$;
$h_2 = p_2 - p_1$;
$\delta_1 = (f(p_1) - f(p_0))/h_1$;
$\delta_2 = (f(p_2) - f(p_1))/h_2$;
$d = (\delta_2 - \delta_1)/(h_2 + h_1)$;
$i = i + 1$.

Step 8 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP. ■

Illustration Consider the polynomial $f(x) = x^4 - 3x^3 + x^2 + x + 1$, part of whose graph is shown in Figure 2.14.

Figure 2.14



Three sets of three initial points will be used with Algorithm 2.8 and $TOL = 10^{-5}$ to approximate the zeros of f. The first set will use $p_0 = 0.5$, $p_1 = -0.5$, and $p_2 = 0$. The parabola passing through these points has complex roots because it does not intersect the x-axis. Table 2.12 gives approximations to the corresponding complex zeros of f.

Table 2.12

	$p_0 = 0.5,\ p_1 = -0.5,\ p_2 = 0$	
i	p_i	$f(p_i)$
3	$-0.100000 + 0.888819i$	$-0.01120000 + 3.014875548i$
4	$-0.492146 + 0.447031i$	$-0.1691201 - 0.7367331502i$
5	$-0.352226 + 0.484132i$	$-0.1786004 + 0.0181872213i$
6	$-0.340229 + 0.443036i$	$0.01197670 - 0.0105562185i$
7	$-0.339095 + 0.446656i$	$-0.0010550 + 0.000387261i$
8	$-0.339093 + 0.446630i$	$0.000000 + 0.000000i$
9	$-0.339093 + 0.446630i$	$0.000000 + 0.000000i$

Table 2.13 gives the approximations to the two real zeros of f. The smallest of these uses $p_0 = 0.5$, $p_1 = 1.0$, and $p_2 = 1.5$, and the largest root is approximated when $p_0 = 1.5$, $p_1 = 2.0$, and $p_2 = 2.5$.

Table 2.13

$p_0 = 0.5,\ p_1 = 1.0,\ p_2 = 1.5$			$p_0 = 1.5,\ p_1 = 2.0,\ p_2 = 2.5$		
i	p_i	$f(p_i)$	i	p_i	$f(p_i)$
3	1.40637	−0.04851	3	2.24733	−0.24507
4	1.38878	0.00174	4	2.28652	−0.01446
5	1.38939	0.00000	5	2.28878	−0.00012
6	1.38939	0.00000	6	2.28880	0.00000
			7	2.28879	0.00000

The values in the tables are accurate approximations to the places listed. □

We used Maple to generate the results in Table 2.12. To find the first result in the table, define $f(x)$ with

$f := x \to x^4 - 3x^3 + x^2 + x + 1$

Then enter the initial approximations with

$p0 := 0.5; p1 := -0.5; p2 := 0.0$

and evaluate the function at these points with

$f0 := f(p0); f1 := f(p1); f2 := f(p2)$

To determine the coefficients a, b, c, and the approximate solution, enter

$c := f2;$

$$b := \frac{\left((p0 - p2)^2 \cdot (f1 - f2) - (p1 - p2)^2 \cdot (f0 - f2)\right)}{(p0 - p2) \cdot (p1 - p2) \cdot (p0 - p1)}$$

$$a := \frac{((p1 - p2) \cdot (f0 - f2) - (p0 - p2) \cdot (f1 - f2))}{(p0 - p2) \cdot (p1 - p2) \cdot (p0 - p1)}$$

$$p3 := p2 - \frac{2c}{b + \left(\frac{b}{abs(b)}\right)\sqrt{b^2 - 4a \cdot c}}$$

This produces the final Maple output

$$-0.1000000000 + 0.8888194418I$$

and evaluating at this approximation gives $f(p3)$ as

$$-0.0112000001 + 3.014875548I$$

This is our first approximation, as seen in Table 2.12.

The illustration shows that Müller's method can approximate the roots of polynomials with a variety of starting values. In fact, Müller's method generally converges to the root of a polynomial for any initial approximation choice, although problems can be constructed for which convergence will not occur. For example, suppose that for some i we have $f(p_i) = f(p_{i+1}) = f(p_{i+2}) \neq 0$. The quadratic equation then reduces to a nonzero constant function and never intersects the x-axis. This is not usually the case, however, and general-purpose software packages using Müller's method request only one initial approximation per root and will even supply this approximation as an option.

EXERCISE SET 2.6

1. Find the approximations to within 10^{-4} to all the real zeros of the following polynomials using Newton's method.

a. $f(x) = x^3 - 2x^2 - 5$

b. $f(x) = x^3 + 3x^2 - 1$

c. $f(x) = x^3 - x - 1$

d. $f(x) = x^4 + 2x^2 - x - 3$

e. $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$

f. $f(x) = x^5 - x^4 + 2x^3 - 3x^2 + x - 4$

2. Find approximations to within 10^{-5} to all the zeros of each of the following polynomials by first finding the real zeros using Newton's method and then reducing to polynomials of lower degree to determine any complex zeros.

a. $f(x) = x^4 + 5x^3 - 9x^2 - 85x - 136$

b. $f(x) = x^4 - 2x^3 - 12x^2 + 16x - 40$

c. $f(x) = x^4 + x^3 + 3x^2 + 2x + 2$

d. $f(x) = x^5 + 11x^4 - 21x^3 - 10x^2 - 21x - 5$

e. $f(x) = 16x^4 + 88x^3 + 159x^2 + 76x - 240$

f. $f(x) = x^4 - 4x^2 - 3x + 5$

g. $f(x) = x^4 - 2x^3 - 4x^2 + 4x + 4$

h. $f(x) = x^3 - 7x^2 + 14x - 6$

3. Repeat Exercise 1 using Müller's method.

4. Repeat Exercise 2 using Müller's method.

5. Use Newton's method to find, within 10^{-3}, the zeros and critical points of the following functions. Use this information to sketch the graph of f.

a. $f(x) = x^3 - 9x^2 + 12$

b. $f(x) = x^4 - 2x^3 - 5x^2 + 12x - 5$

6. $f(x) = 10x^3 - 8.3x^2 + 2.295x - 0.21141 = 0$ has a root at $x = 0.29$. Use Newton's method with an initial approximation $x_0 = 0.28$ to attempt to find this root. Explain what happens.

7. Use Maple to find a real zero of the polynomial $f(x) = x^3 + 4x - 4$.

8. Use Maple to find a real zero of the polynomial $f(x) = x^3 - 2x - 5$.

9. Use each of the following methods to find a solution in $[0.1, 1]$ accurate to within 10^{-4} for

$$600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0.$$

a. Bisection method

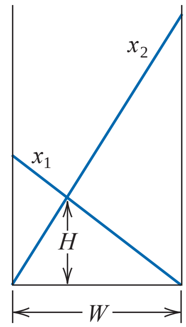
b. Newton's method

c. Secant method

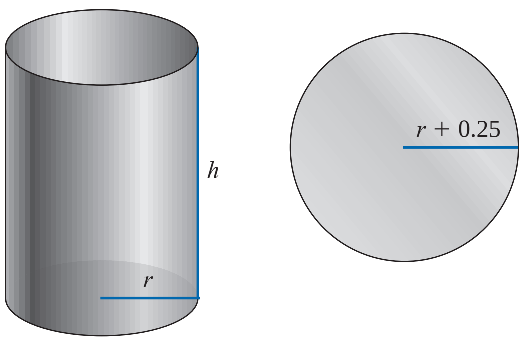
d. method of False Position

e. Müller's method

10. Two ladders crisscross an alley of width W. Each ladder reaches from the base of one wall to some point on the opposite wall. The ladders cross at a height H above the pavement. Find W given that the lengths of the ladders are $x_1 = 20$ ft and $x_2 = 30$ ft, and that $H = 8$ ft.



11. A can in the shape of a right circular cylinder is to be constructed to contain 1000 cm^3. The circular top and bottom of the can must have a radius of 0.25 cm more than the radius of the can so that the excess can be used to form a seal with the side. The sheet of material being formed into the side of the can must also be 0.25 cm longer than the circumference of the can so that a seal can be formed. Find, to within 10^{-4}, the minimal amount of material needed to construct the can.



12. In 1224, Leonardo of Pisa, better known as Fibonacci, answered a mathematical challenge of John of Palermo in the presence of Emperor Frederick II: find a root of the equation $x^3 + 2x^2 + 10x = 20$. He first showed that the equation had no rational roots and no Euclidean irrational root—that is, no root in any of the forms $a \pm \sqrt{b}$, $\sqrt{a} \pm \sqrt{b}$, $\sqrt{a \pm \sqrt{b}}$, or $\sqrt{\sqrt{a} \pm \sqrt{b}}$, where a and b are rational numbers. He then approximated the only real root, probably using an algebraic technique of Omar Khayyam involving the intersection of a circle and a parabola. His answer was given in the base-60 number system as

$$1 + 22\left(\frac{1}{60}\right) + 7\left(\frac{1}{60}\right)^2 + 42\left(\frac{1}{60}\right)^3 + 33\left(\frac{1}{60}\right)^4 + 4\left(\frac{1}{60}\right)^5 + 40\left(\frac{1}{60}\right)^6.$$

How accurate was his approximation?