Chapter4: Probability and counting rule

Basic Concepts:

Set: is a frame contains an elements. Denoted by capital letter as $\boldsymbol{A}, \boldsymbol{B}, \ldots$

Element: is the smallest content of a set. Denoted by small letter as $\boldsymbol{a}, \boldsymbol{b}, \ldots$

Empty set: is a set that not have any element. Denoted by $\boldsymbol{\Phi} = \{\}$.

The **Union**: of two sets is a new set that contains all of the elements that are in **at least** one of the two sets. The union is written as $\boldsymbol{A} \cup \boldsymbol{B}$ or "$\boldsymbol{A}$ **or** $\boldsymbol{B}$".

The **intersection:** of two sets is a new set that contains all of the elements that are in **both** sets. The intersection is written as $\boldsymbol{A} \cap \boldsymbol{B}$ or "$\boldsymbol{A}$ **and** $\boldsymbol{B}$".

Symbol ($\in$): Denoted to an Element **Belongs** to a set. a $\in \boldsymbol{A}$
Symbol ($\subset$): Denoted to an set as a subset of another set.B $\subset \boldsymbol{A}$

A **sample space ($\boldsymbol{\Omega}$):** is the set of all possible outcomes of a probability experiment.

Event: is a subset of sample space (Ω).

Equality of set: Set A = Set B if all contains same elements.

The **complement** of a set A, often denoted by A^c or $\bar{A}$, are the elements not in A.
$(\bar{A} = \Omega - A)$, $\bar{A} \cup \boldsymbol{A} = \Omega$ and $\bar{A} \cap \boldsymbol{A} = \boldsymbol{\Phi}$

Disjoint events are events that never occur at the same time. These are also known as **mutually exclusive events**. Note: $\boldsymbol{A}$ and $\bar{A}$ are **disjoint . if A and B are disjoint**: $A \cap \boldsymbol{B} = \boldsymbol{\Phi}$

Independent events do not affect one another and do not increase or decrease the probability of another event happening.

An **outcome** is the result of a single trial of a probability experiment.

Probability as a general concept can be defined as the chance of an event occurring.

Example: For an experiment of rolling a die one-time, find:

A) The sample space of the experiment: $\Omega = \{1, 2, 3, 4, 5, 6\}$

B) Set A, which contains the odd numbers: $A = \{1, 3, 5\}$

C) Set B, which contains the even numbers: $B = \{2, 4, 6\}$

D) Set C containing numbers less than 5 for the experiment: $C = \{1, 2, 3, 4\}$

E) A∩B: $A \cap B = \{\} = \phi$, is disjoint

F) A∩C: $A \cap C = \{1, 3\}$

G) A∪C: $A \cup C = \{1, 2, 3, 4, 5\}$

H) Complement of set A: $\bar{A} = \{2, 4, 6\} = B$

I) Set D that contains numbers can divided by 3: $D = \{3, 6\}$

II) Set of $\Omega - C$: $\Omega - C = \{5, 6\} = \bar{C}$

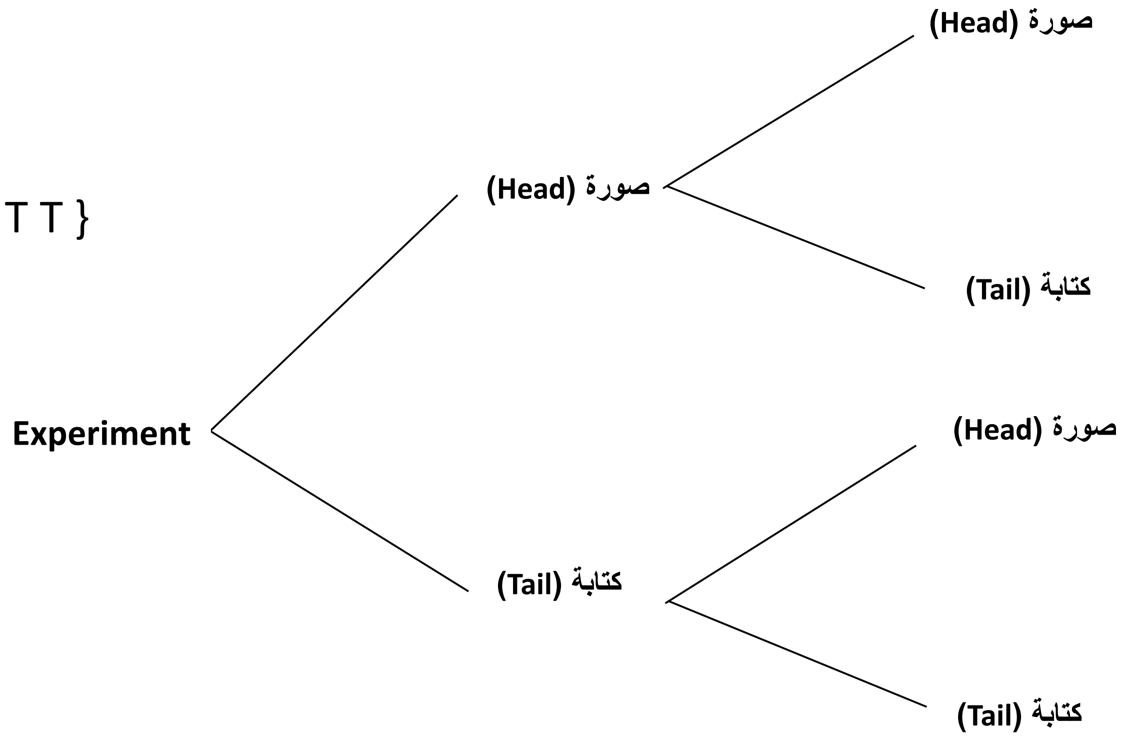
III) Set A - C (i.e. the set of A elements excluding all C elements): $A - C = \{5\}$

Tree diagram:

For the experiment of tossing a coin 2 times

a) **Draw a tree diagram :**

b) **Find the sample space:** Ω={ H H, H T, T H, T T }



Counting Rules:

1. **The multiplicative rule:**

   if we can conduct an experiment with several stages, where by:
   If the experiment at first stage done with n_1 methods
   If the experiment at second stage done with n_2 methods
   ….
   If the experiment at k stage done with n_k methods

   The number of ways to do the experiment as a whole (in all its stages from 1 to K) is $\boldsymbol{n_1 * n_2 * \ldots * n_k}$

2. **The additive rule:**

   if we can conduct an experiment with several stages, where by:
   If the experiment at first stage done with n_1 methods
   If the experiment at second stage done with n_2 methods
   ….
   If the experiment at k stage done with n_k methods
   And All these stages are mutually exclusive events

   The number of ways to do the experiment as a whole (in all its stages from 1 to K) is $\boldsymbol{n_1 + n_2 + \ldots + n_k}$

3. **The combination rule:**

   Is the number of ways to select a subset of (n) elements from a set of (N) elements, without regard to the order in which they were selected. (Note: $\boldsymbol{n \leq N}$) is:

$$_N\mathbb{C}_n = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

4. The permutations rule:

a) **Is the number of ways to arrange a group of N different elements in a set of n places (the order is important). (Note: $n \leq N$) is:**

$$ {}_{N}\mathbb{P}_{n} = \frac{N!}{(N-n)!} $$

b) Is the number of ways to arrange a group of N elements (some of elements are repeated more than one time) in a set of N places (the order is important). (Note: N $= n_1 + n_2 + \cdots + n_k$) is:

$$ \frac{N!}{n_1!n_2!\ldots n_k!} $$

$$ \boldsymbol{Probability\ of\ event\ E = \frac{\#\ of\ elements\ in\ E}{\#\ of\ elements\ in\ \Omega}} $$

$$ \boldsymbol{P(E) = \frac{\#\ of\ ways\ to\ do\ the\ experiment\ of\ event\ (E)}{\#\ of\ ways\ to\ do\ the\ experiment\ at\ all\ (\Omega)}} $$

Example: For an experiment of rolling a die one-time, find:

A) The sample space of the experiment: $\Omega = \{1\ ,\ 2\ ,\ 3\ ,\ 4\ ,\ 5\ ,\ 6\}$

B) Probability of Set A, which contains the odd numbers: $A = \{1\ ,\ 3\ ,\ 5\}$

C) Probability of Set B, which contains the even numbers: $B = \{2\ ,\ 4\ ,\ 6\}$

D) Probability of Set C containing numbers less than 5 for the experiment: $C = \{1\ ,\ 2\ ,\ 3\ ,\ 4\}$

E) Probability of A∩B: $A \cap B = \{\,\} = \phi$, is disjoint

F) Probability of A∩C: $A \cap C = \{1\ ,\ 3\}$

G) Probability of A∪C: $A \cup C = \{1\ ,\ 2\ ,\ 3\ ,\ 4\ ,\ 5\}$

H) Probability of Complement of set A: $\bar{A} = \{\ 2\ ,\ 4\ ,\ 6\} =$ B

I) Probability of Set D that contains numbers can divided by 3: $D = \{3\ ,\ \ 6\}$

II) Probability of Set of $\Omega - C$: $\Omega - C = \{\ 5\ ,\ 6\ \} = \bar{\mathrm{C}}$

III) Probability of Set (A – C) (i.e. the set of A elements excluding all C elements): $A - C = \{\ 5\ \}$

Important probability definitions:

1. If all events A_i *are disjoint* $P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + \cdots + P(A_k)$

2. The probability of occurrence of two Events A&B together means (intersection).

3. The probability of occurrence Events A or occurrence Events B
OR the probability of occurrence at least one of two Events A , B is (union).

4. The Events A and B are independent iff $P(A \cap B) = P(A)\,P(B)$
occurrence any Events not affects on other one.

5. The two events are disjoint (mutually exclusive) iff $P(A \cap B) = P(\phi) = 0$

6. The two events are exhaustive mutually exclusive iff
$P(A \cap B) = P(\phi) = 0$ (disjoint) &&& $P(A \cup B) = P(\Omega) = 1$

7. If event A is subset of event B. ($A \subset B$) then: $P(A \cap B) = P(A)$, $P(A \cup B) = P(B)$, $P(A) \leq P(B)$

Probability Rules

	Probability of
$\boldsymbol{P(\bar{A}) = 1 - P(A)}$	
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$	
$P(\bar{A} \cap B) = P(B - \mathrm{A}) = P(B) - P(A \cap B)$	B occurrence and A not occurrence (B occurring except A)
$P(A \cap \bar{B}) = P(\mathrm{A} - B) = P(A) - P(A \cap B)$	A occurrence and B not occurrence (A occurrence except B)
$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$	A not occurrence AND B not occurrence (Either it will not occurring)
$P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$	A not occurrence OR B not occurrence (They do not occurring together) (occurrence one of them at most)

Example: How many different SIM cards can be issued in the market if each of them consists of 10 digits and each chip starts from the left with (059)

1	2	3	4	5	6	7	8	9	10	Cell order
0	5	9	0, 1, 2, 3, 4, 5, 6, 7, 8, ,9	0, 1, 2, 3, 4, 5, 6, 7, 8, ,9	0, 1, 2, 3, 4, 5, 6, 7, 8, ,9	0, 1, 2, 3, 4, 5, 6, 7, 8, ,9	0, 1, 2, 3, 4, 5, 6, 7, 8, ,9	0, 1, 2, 3, 4, 5, 6, 7, 8, ,9	0, 1, 2, 3, 4, 5, 6, 7, 8, 9,	Possible number can use
# of ways	# of ways	# of ways	# of ways	# of ways	# of ways	# of ways	# of ways	# of ways	# of ways	can be issued 10 Million SIM
1	1	1	10	10	10	10	10	10	10	

1 * 1 * 1 * 10 * 10 * 10 * 10 * 10 * 10 * 10 = **10 000 000**

Example: **How many different words consisting of 3 letters (without repeating the letter and regardless of its meaning) can be formed from the letters of the word (write)?**

Use permutations or multiplication

The solution:

we want to arrange 5 different letters} characters in the word (write) {in 3 places (cells),

so (we use permutations / rule A $\frac{5!}{(5-3)!}$ **= 60 different words**

Example: **How much methods can you rearrange the letters of a word (statistics) to form 10-letter words that do not necessarily have meaning?**

Solution:

We want to arrange 10 letters, some of them are repeated in 10 places (cells), so (we use permutations / the B rule)

Number of times the letter S = n1 = 3

Number of times the letter T = n2 = 3

Number of times the letter A = n3 = 1

Number of times the letter I = n4 = 2

Number of times the letter C = n5 = 1

$\frac{10!}{3!\ 3!\ 1!\ 2!\ 1!}$ **= 50400 methods**

where N = n1 + n2 +… + n5 = 10

Example:

How many ways can stand 8 students in a queue?

Solution:

We want to arrange 8 different elements into 8 places, so (we use permutations / rule A)

$$\frac{8!}{(8-8)!} = 40320$$

This example can be solved using the multiplication rule: 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1 = 40320

Example:

A student wants to register two courses from available courses offered for this semester, which were (3 core courses) and (5 elective courses).

How many ways can the two courses be registered (the number of options available for registration), provided that the two courses are of the same type (core or elective).

Solution:

Note that registering core courses prevents registering the elective courses (so we use the addition rule)

We want to register 2 core courses from the 3 core courses (regardless of order) (we use combinations)

Number of ways to register the core courses is:

$$n_1 = \binom{3}{2} = 3$$

We want to register 2 elective courses from the 5 elective courses (regardless of order) (we use combinations)

Number of ways to register the elective courses is:

$$n_2 = \binom{5}{2} = 10$$

Number of ways to conduct the experiment as a whole is n1 + n2 = 3 + 10 = 13 ways can the two courses be registered

(Note that we used the addition rule because the occurrence (registration) of first course type prevents the occurrence (registration) of the others)

Example: **How many different words consisting of 4 letters (without repeating the letter and regardless of its meaning) can be formed from the letters of the word (find)?**

Use permutations or multiplication

The solution:

we want to arrange 4 different letters} characters in the word (**find**) {in 4 places (cells),

$$\frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{4!}{1} = 4 * 3 * 2 * 1 = 24 \text{ different words}$$

Example: **How many different car cards can be issued in the market if each of them consists of 6 digits and each card starts from the left with (18), and then two letters from the words letter (Find), and two numbers**

1	2	3	4	5	6					Cell order
1	8	f , i , n , d	f , i , n , d	0, 1, 2, 3, 4, 5, 6, 7, 8, ,9	0, 1, 2, 3, 4, 5, 6, 7, 8, ,9					Possible number can use
# of ways	# of ways	# of ways	# of ways	# of ways	# of ways					1600 different car cards
1	1	4	4	10	10					
1 * 1 * 4 * 4 * 10 *10 = 1600										

Example: **Assuming** $\boldsymbol{P(A)} = 0.5$, $\boldsymbol{P(B)} = 0.3$

Compute:

a) $\boldsymbol{P(A \cup B)}$

b) $\boldsymbol{P(A \cap \bar{B})}$

Based on the following cases:

1) A and B are independent

Answer:

$\boldsymbol{P(A \cup B) =}$

$=P(A) + P(B) - P(A \cap B)$

$=P(A) + P(B) - \big(P(A)P(B)\big)$

$=0.5 + 0.3 - 0.15 = 0.65$

$\boldsymbol{P(A \cap \bar{B})} =$

$=P(A) - P(A \cap B)$

$=P(A) - \big(P(A)P(B)\big)$

$=0.5 - 0.15 = 0.35$

2) A and B are disjoint (mutually exclusive)

Answer:

$\boldsymbol{P(A \cup B) =}$

$=P(A) + P(B) - P(A \cap B)$

$=P(A) + P(B) - (0)$

$=0.5 + 0.3 - 0 = 0.8$

$\boldsymbol{P(A \cap \bar{B})} =$

$=P(A) - P(A \cap B)$

$=P(A) - (0) = 0.5 - 0 = 0.5$

3) B is a subset of A

Answer:

$\boldsymbol{P(A \cup B) =}$

$=P(A) + P(B) - P(A \cap B)$

$=P(A) + P(B) - \big(P(B)\big)$

$=P(A) = 0.5$

$\boldsymbol{P(A \cap \bar{B})} =$

$=P(A) - P(A \cap B)$

$=P(A) - \big(P(B)\big)$

$=0.5 - 0.3 = 0.2$

Conditional probability: **Bayes' theorem**

Definition: The probability of event A occurs conditional that (if you know that) event B is already occurred

$$P\ (A \mid B)\ =\ \frac{P\ (A \cap B)}{P\ (B)}$$

Definition: The probability of event B occurs conditional that (if you know that) event A is already occurred

$$P\ (B \mid A)\ =\ \frac{P\ (A \cap B)}{P\ (A)}$$

Example: if $\boldsymbol{P(A) = 0.6}$, $\boldsymbol{P(B) = 0.3}$, $\boldsymbol{P(A \cap B) = 0.1}$, find

1) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.3} = \frac{1}{3}$

2) $P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{0.6 - 0.1}{1 - 0.3} = \frac{0.5}{0.7} = \frac{5}{7}$

Example: For an experiment of tossing a die Two-time, compute the probability of:

1) The two faces are equal (event A)

Where: A = {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)}

$$P\,(A) = \frac{\#\,of\ elements\ in\ A}{\#\,of\ elements\ in\ \Omega} = \frac{6}{36}$$

2) The two faces are different (event B) (same as $\bar{A}$)

$$P\,(B) = \frac{\#\,of\ elements\ in\ B}{\#\,of\ elements\ in\ \Omega} = \frac{30}{36}$$

3) The summation of the two faces equals 10 (event C): C = { (5,5) , (4,6) , (6,4) }

$$P\,(C) = \frac{\#\,of\ elements\ in\ C}{\#\,of\ elements\ in\ \Omega} = \frac{3}{36}$$

4) The summation of the two faces is greater than 10 (event D): D = { (5,6) , (6,5) , (6,6) }

$$P\,(D) = \frac{\#\,of\ elements\ in\ D}{\#\,of\ elements\ in\ \Omega} = \frac{3}{36}$$

5) The two faces are equal (event A) **provided that** the first number is odd (event E)

The two faces are equal (event A) **if** the first number is odd (event E)

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)} = \frac{(3/36)}{(18/36)} = \frac{3}{18}$$

6) The summation of the two faces equals 10 (event C) **provided that** the first number is odd (event E)

$$P(C \mid E) = \frac{P(C \cap E)}{P(E)} = \frac{(1/36)}{(18/36)} = \frac{1}{18}$$

7) The summation of the two faces is greater than 10 (event D) **provided that** the first number is even (event F)

$$P(D \mid F) = \frac{P(D \cap F)}{P(F)} = \frac{(2/36)}{(18/36)} = \frac{2}{18}$$

Example: A box contains 4 red balls (R) and 3 white balls (W).

If we withdraw two balls from the box with replacement, find the probability:

A) The first ball is red and the second white

B) The two balls are red

C) The two balls are white

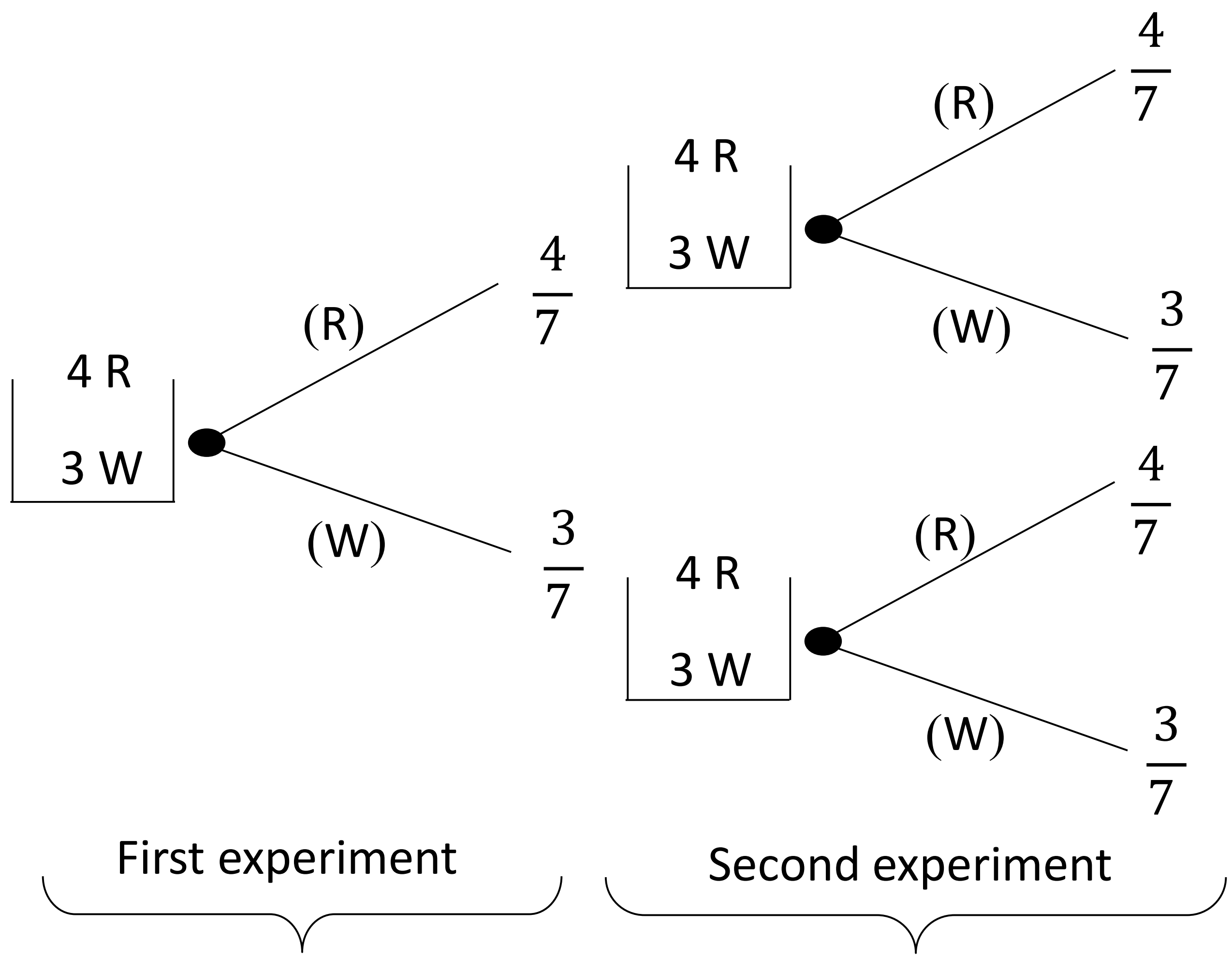
D) The two balls are of same color

E) The two balls are of different color

Note: the first experiment does not affect on the second experiment because the withdrawal is with replacement. Therefore we will use the independent events laws.

i.e. The word (and) is equivalent to the intersection, the intersection of two independent events is:

$P(A \text{ and } B) = P(A \cap B) = P(A)\, P(B)$



A) The first ball is red and the second white

$$P\left((1^{st}\text{ is red}) \cap \left(2^{nd}\text{ is white}\right)\right)$$

$$= P(1^{st}\text{ is red})\, P\left(2^{nd}\text{ is white}\right) = \frac{4}{7}\,\frac{3}{7} = \frac{12}{49}$$

B) The two balls are red

$$P\left((1^{st}\text{ is red}) \cap \left(2^{nd}\text{ is red}\right)\right)$$

$$= P(1^{st}\text{ is red})\, P\left(2^{nd}\text{ is red}\right) = \frac{4}{7}\,\frac{4}{7} = \frac{16}{49}$$

C) The two balls are white

$$P\left((1^{st}\text{ is white}) \cap \left(2^{nd}\text{ is white}\right)\right)$$

$$= P(1^{st}\text{ is white})\, P\left(2^{nd}\text{ is white}\right) = \frac{3}{7}\,\frac{3}{7} = \frac{9}{49}$$

D) The two balls are of same color (disjoint events) : event E

The two balls are red OR The two balls are white

$$\frac{16}{49}+\frac{9}{49} - 0 = \frac{25}{49}$$

E) The two balls are of different color ($\bar{E}$)

$$1-\frac{25}{49} = \frac{24}{49}$$

Example: A box contains 4 red balls (R) and 3 white balls (W).

If we withdraw two balls from the box without replacement, find the probability:

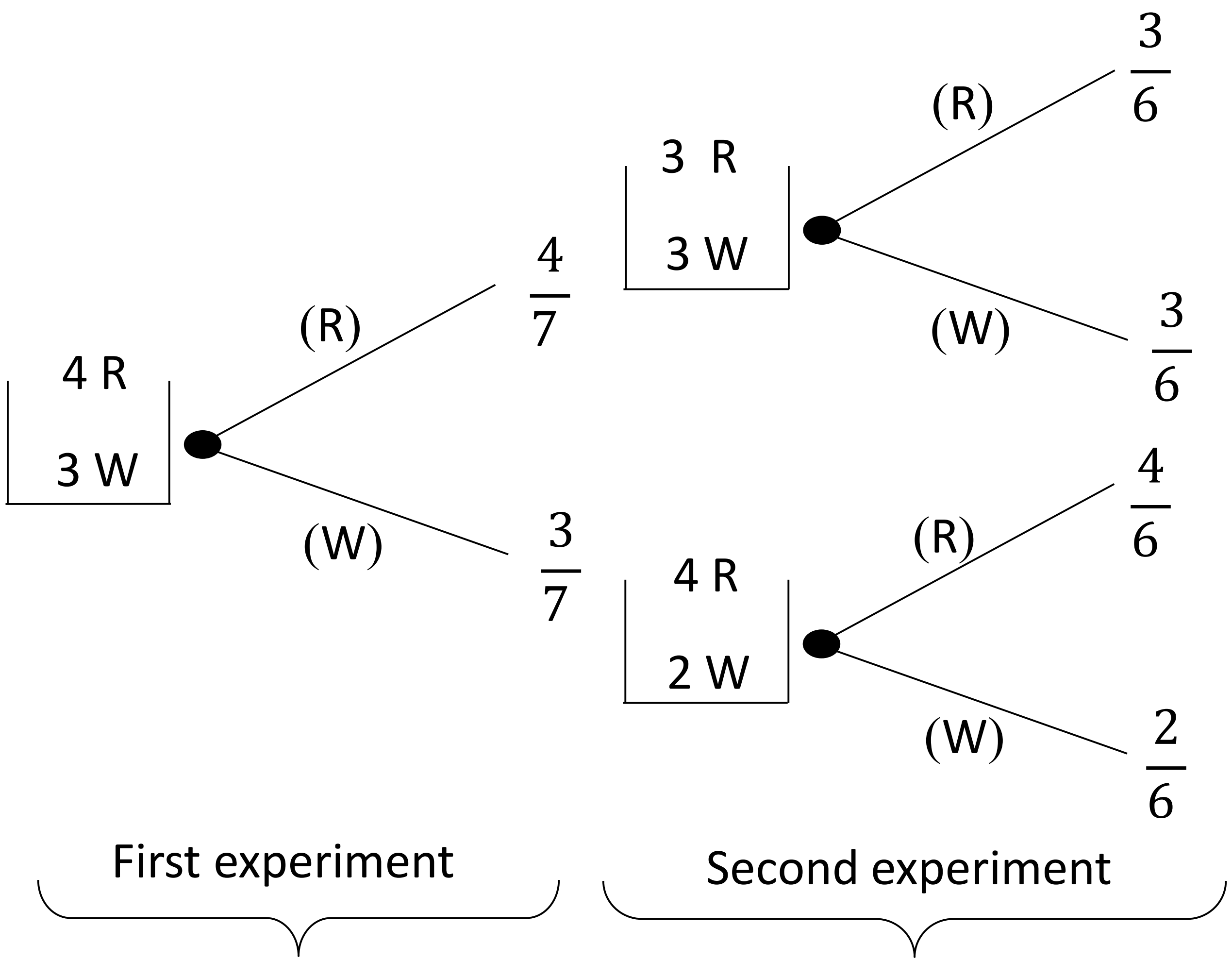
A) The first ball is red and the second white

B) The two balls are red

C) The two balls are white

D) The two balls are of same color

E) The two balls are of different color



A) The first ball is red and the second white

$$P\Big((1^{\text{st}}\text{ is red}) \cap \big(2^{\text{nd}}\text{ is white}\big)\Big)$$

$$= P(1^{\text{st}}\text{ is red})\, P\big(2^{\text{nd}}\text{ is white}\big) = \frac{4}{7}\,\frac{3}{6} = \frac{12}{42}$$

B) The two balls are red

$$P\Big((1^{\text{st}}\text{ is red}) \cap \big(2^{\text{nd}}\text{ is red}\big)\Big)$$

$$= P(1^{\text{st}}\text{ is red})\, P\big(2^{\text{nd}}\text{ is red}\big) = \frac{4}{7}\,\frac{3}{6} = \frac{12}{42}$$

C) The two balls are white

$$P\Big((1^{\text{st}}\text{ is white}) \cap \big(2^{\text{nd}}\text{ is white}\big)\Big)$$

$$= P(1^{\text{st}}\text{ is white})\, P\big(2^{\text{nd}}\text{ is white}\big) = \frac{3}{7}\,\frac{2}{6} = \frac{6}{42}$$

D) The two balls are of same color (disjoint events) : event E

The two balls are red OR The two balls are white

$$\frac{12}{42} + \frac{6}{42} - 0 = \frac{18}{42}$$

E) The two balls are of different color ($\bar{E}$)

$$1 - \frac{18}{42} = \frac{24}{42}$$

Example: A box contains 4 red balls (R) and 3 white balls (W).

If we withdraw two balls from the box (draw together), find the probability:

A) The two balls are red

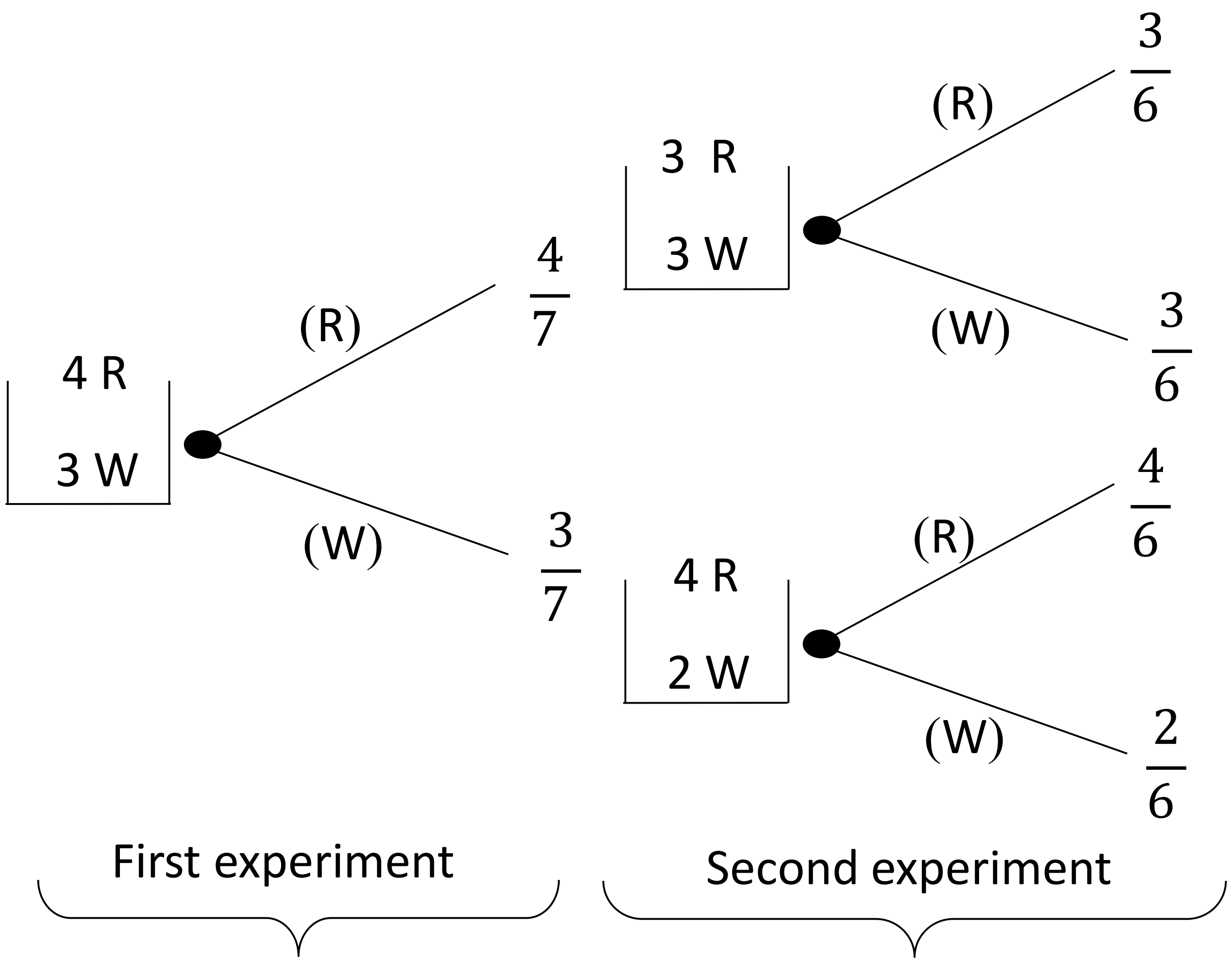
B) The two balls are white

C) The two balls are of same color

D) The two balls are of different color

Note: Same as the tree of without replacement *** You can use **the combination rule**

Same as the tree of without replacement



A) The two balls are red

$P\left((1^{st}\text{ is red})\ \cap (2^{nd}\text{ is red})\right)$

$= P(1^{st}\text{ is red})\, P(2^{nd}\text{ is red}) = \frac{4}{7}\,\frac{3}{6} = \frac{12}{42} = \frac{6}{21}$

OR $P\left((2\text{ R})\ \cap (0\text{ W})\right) = \frac{\binom{4}{2}\binom{3}{0}}{\binom{7}{2}} = \frac{6*1}{21} = \frac{6}{21}$

B) The two balls are white

$P\left((1^{st}\text{ is white})\ \cap (2^{nd}\text{ is white})\right)$

$= P(1^{st}\text{ is white})\, P(2^{nd}\text{ is white}) = \frac{3}{7}\,\frac{2}{6} = \frac{6}{42}$

$= \frac{3}{21}$

OR $P\left((0\text{ R})\ \cap (2\text{ W})\right) = \frac{\binom{4}{0}\binom{3}{2}}{\binom{7}{2}} = \frac{1*3}{21} = \frac{3}{21}$

C) The two balls are of same color (disjoint events) : event E

The two balls are red OR The two balls are white

$\frac{12}{42} + \frac{6}{42} - 0 = \frac{18}{42}$

D) The two balls are of different color ($\bar{E}$)

$1 - \frac{18}{42} = \frac{24}{42}$