Chapter5: Probability distributions

A random variable is a variable whose values are determined by chance.

Note: classify variables as discrete or continuous by observing the values the variable can assume.

Discrete variable: If a variable values is countable (can assume only a specific number of values).

Example:
the outcomes for the roll of a die, the outcomes for the toss of a coin, the number of children in a family, the number of cars in a road.

Continuous variable: If a variable values is Uncountable (usually contains fractions).
Variables that can assume all values in the interval between any two given values are called continuous variables.

Example:
the body temperature, the body weight , the body height.

Probability Distribution of a random variable (r.v.):

is a table, rule, or function (formula) describes all possible probabilities of r.v.

Example:

Obtain the probability distribution of the variable: Number of heads in the experiment of tossing a coin 3 times?

Answer:

The sample space of our experiment is

$\Omega = \{ TTT, \quad TTH, \quad THT, \quad HTT, \quad HHT, \quad HTH, \quad THH, \quad HHH \}$

Assuming X is the random variable for (the number of heads), then X assumes the value 0, 1, 2, or 3.

The random variable X is discrete variable since it is countable. Then, the probability distribution of X is:

Probabilities for the values of X can be determined as follows:

No heads	One head			Two heads			Three heads
TTT	TTH	THT	HTT	HHT	HTH	THH	HHH
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\frac{1}{8}$	$\frac{3}{8}$			$\frac{3}{8}$			$\frac{1}{8}$

X: Number of heads	0	1	2	3
Probability of x: P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Assuming a discrete random variable X, then:

Two Requirements for a Probability Distribution

1. The sum of the probabilities of all the events in the sample space must equal 1; that is, $\Sigma P(X) = 1$.
2. The probability of each event in the sample space must be between or equal to 0 and 1. That is, $0 \leq P(X) \leq 1$.

Example:

Obtain the probability distribution of the variable (the faced number) in the experiment of rolling a die one time.

Solution

Since the sample space is 1, 2, 3, 4, 5, 6 and each outcome has a probability of $\frac{1}{6}$, the distribution is as shown.

Outcome X	1	2	3	4	5	6
Probability $P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Probability Distributions

Example: Determine whether each distribution is a probability distribution.

a.

X	0	5	10	15	20
$P(X)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

b.

X	0	2	4	6
$P(X)$	-1.0	1.5	0.3	0.2

c.

X	1	2	3	4
$P(X)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{9}{16}$

d.

X	2	3	7
$P(X)$	0.5	0.3	0.4

Solution

a. Yes, it is a probability distribution.

b. No, it is not a probability distribution, since $P(X)$ cannot be 1.5 or -1.0.

c. Yes, it is a probability distribution.

d. No, it is not, since $\Sigma P(X) = 1.2$.

Formula for the Mean of a Probability Distribution

The mean of a random variable with a discrete probability distribution is

$$\mu = X_1 \cdot P(X_1) + X_2 \cdot P(X_2) + X_3 \cdot P(X_3) + \cdots + X_n \cdot P(X_n)$$
$$= \Sigma X \cdot P(X)$$

where $X_1, X_2, X_3, \ldots, X_n$ are the outcomes and $P(X_1), P(X_2), P(X_3), \ldots, P(X_n)$ are the corresponding probabilities.

Note: $\Sigma X \cdot P(X)$ means to sum the products.

Formula for the Variance of a Probability Distribution

Find the variance of a probability distribution by multiplying the square of each outcome by its corresponding probability, summing those products, and subtracting the square of the mean. The formula for the variance of a probability distribution is

$$\sigma^2 = \Sigma[X^2 \cdot P(X)] - \mu^2$$

The standard deviation of a probability distribution is

$$\sigma = \sqrt{\sigma^2} \qquad \text{or} \qquad \sqrt{\Sigma[X^2 \cdot P(X)] - \mu^2}$$

The expected value of a discrete random variable of a probability distribution is the theoretical average of the variable. The formula is $E(X) = \mu = \sum X \,.\, P(X)$

Example 5–11 On Hold for Talk Radio

A talk radio station has four telephone lines. If the host is unable to talk (i.e., during a commercial) or is talking to a person, the other callers are placed on hold. When all lines are in use, others who are trying to call in get a busy signal. The probability that 0, 1, 2, 3, or 4 people will get through is shown in the distribution. Find the variance and standard deviation for the distribution.

X	0	1	2	3	4
$P(X)$	0.18	0.34	0.23	0.21	0.04

Should the station have considered getting more phone lines installed?

Solution

The mean is

$$\mu = \Sigma X \cdot P(X)$$
$$= 0 \cdot (0.18) + 1 \cdot (0.34) + 2 \cdot (0.23) + 3 \cdot (0.21) + 4 \cdot (0.04)$$
$$= 1.6$$

The variance is

$$\sigma^2 = \Sigma[X^2 \cdot P(X)] - \mu^2$$
$$= [0^2 \cdot (0.18) + 1^2 \cdot (0.34) + 2^2 \cdot (0.23) + 3^2 \cdot (0.21) + 4^2 \cdot (0.04)] - 1.6^2$$
$$= [0 + 0.34 + 0.92 + 1.89 + 0.64] - 2.56$$
$$= 3.79 - 2.56 = 1.23$$
$$= 1.2 \text{ (rounded)}$$

The standard deviation is $\sigma = \sqrt{\sigma^2}$, or $\sigma = \sqrt{1.2} = 1.1$.

The Binomial Distribution

Many types of probability problems have only two outcomes or can be reduced to two outcomes.

For example,

when a coin is tossed, it can land heads or tails.

When a baby is born, it will be either male or female.

In a basketball game, a team either wins or loses.

A true/false item can be answered in only two ways, true or false.

A **binomial experiment** is a probability experiment that satisfies the following four requirements:

1. There must be a fixed number of trials.
2. Each trial can have only two outcomes or outcomes that can be reduced to two outcomes. These outcomes can be considered as either success or failure.
3. The outcomes of each trial must be independent of one another.
4. The probability of a success must remain the same for each trial.

A binomial experiment and its results give rise to a special probability distribution called the *binomial distribution*.

The outcomes of a binomial experiment and the corresponding probabilities of these outcomes are called a **binomial distribution.**

Notation for the Binomial Distribution

$P(S)$	The symbol for the probability of success
$P(F)$	The symbol for the probability of failure
p	The numerical probability of a success
q	The numerical probability of a failure

$$P(S) = p \qquad \text{and} \qquad P(F) = 1 - p = q$$

n	The number of trials
X	The number of successes in n trials

Note that $0 \leq X \leq n$ and $X = 0, 1, 2, 3, \ldots, n$.

The probability of a success in a binomial experiment can be computed with this formula.

Binomial Probability Formula

In a binomial experiment, the probability of exactly X successes in n trials is

$$P(X) = \frac{n!}{(n-X)!X!} \cdot p^X \cdot q^{n-X}$$

Example:

Obtain the probability distribution of the variable: Number of heads in the experiment of tossing a coin 3 times?

Answer: The sample space of our experiment is

$$\Omega = \{ TTT, \quad TTH, \quad THT, \quad HTT, \quad HHT, \quad HTH, \quad THH, \quad HHH \}$$

Assuming X is the random variable for (the number of heads), then X assumes the value 0, 1, 2, or 3.

The random variable X is discrete variable since it is countable. Then, the probability distribution of X is:

X: Number of heads	0	1	2	3
Probability of x: P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

We can solve this example based on binomial distribution:

The whole experiment consists of 3 **independent** experiment each have only two choices (**success=head** / fail=tail).

Assuming the **success** in this experiment is the (**head**) with probability of (**p=0.5**) and we repeated (tossing a coin) 3 times >>> **n=3**

Then, the random variable X: represents number of heads in the whole experiment: X: 0, 1, 2, and 3

X: is binomial distribution. **p=0.5** and **n=3:**

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \quad , x = 0, 1, 2, \ldots, n$$

$$where: q = (1 - p) \quad , \binom{n}{x} = \frac{n!}{x!(n-x)!}$$

Note:

$n! = n * (n-1) * (n-2) * \cdots * 1$

$n! = n * (n-1) * (n-2)!$

$n! = n * (n-1)!$

$0! = 1$

$1! = 1$

X: is binomial distribution. $p = 0.5$, $q = (0.5) = (1 - p)$ and $n = 3$:

$$P(X = x) = \binom{n}{x} p^x\, q^{n-x} \;, \qquad q = (1 - p) \;, \binom{n}{x} = \frac{n!}{x!\,(n - x)!}$$

$$P(X = x) = \binom{3}{x} 0.5^x\, 0.5^{3-x} \quad ,, \;\; x = 0, 1, 2, 3$$

$$P(X = 1) = \binom{3}{1} 0.5^1\, 0.5^{3-1} = \frac{3!}{1!\,(3 - 1)!} 0.5^1\, 0.5^{3-1} = \frac{3!}{1!\,(2)!} 0.5^1\, 0.5^2 = 3 * 0.5 * 0.25 = 0.375 = \frac{3}{8}$$

$$P(X = 2) = \binom{3}{2} 0.5^2\, 0.5^{3-2} = \frac{3!}{2!\,(3 - 2)!} 0.5^2\, 0.5^{3-2} = \frac{3!}{2!\,(1)!} 0.5^2\, 0.5^1 = 3 * 0.25 * 0.5 = 0.375 = \frac{3}{8}$$

$$P(X = 3) = \binom{3}{3} 0.5^3\, 0.5^{3-3} = \frac{3!}{3!\,(3 - 3)!} 0.5^3\, 0.5^{3-3} = \frac{3!}{3!\,(0)!} 0.5^3\, 0.5^0 = 1 * 0.125 * 1 = 0.125 = \frac{1}{8}$$

$$P(X = 0) = \binom{3}{0} 0.5^0\, 0.5^{3-0} = \frac{3!}{0!\,(3 - 0)!} 0.5^0\, 0.5^{3-0} = \frac{3!}{0!\,(3)!} 0.5^0\, 0.5^3 = 1 * 1 * 0.125 = 0.125 = \frac{1}{8}$$

X: Number of heads	0	1	2	3
Probability of x: P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Mean (Expected value of X) Variance Standard deviation

if X: has a binomial distribution

Expected value of X $$E(X) = mean = \mu = n * p$$

Variance of X $$Var(X) = \sigma^2 = n * p * q$$

Standard deviation of X $$\text{Standard deviation} = \sigma = \sqrt{n * p * q}$$

Example:

Assuming the experiment of tossing a coin **8 times,**

a) Obtain the **probability distribution** of the variable: **Number of heads**?

b) Obtain the probability of getting 5 heads.

c) Obtain the probability of getting at least 7 heads.

d) Obtain the probability of getting 6 heads at most.

e) find E(X) & Var(X)

Solve:

a) Assuming the **success** in this experiment is the (**head**) with probability of (**p=0.5**) and we repeated (tossing a coin) 8 times >>> $n = 8$

X: is **binomial distribution**. $p = 0.5$, $q = (0.5) = (1 - p)$ and $n = 8$: $P(X = x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, 2, \dots, n$

$$P(X = x) = \binom{8}{x} 0.5^x \, 0.5^{8-x} \quad , \qquad x = 0, 1, 2, \dots, 8$$

b) $P(X = 5) = \binom{8}{5} 0.5^5 \, 0.5^{8-5} = 56 * 0.03125 * 0.125 = 0.21875$

c) $P(X \geq 7) = P(X = 7) + P(X = 8) = \binom{8}{7} 0.5^7 \, 0.5^{8-7} + \binom{8}{8} 0.5^8 \, 0.5^{8-8} = 0.03125 + 0.00390625 = 0.0351563$

d) $P(X \leq 6) = 1 - P(X > 6) = 1 - P(X \geq 7) = 1 - 0.0351563 = 0.9648437$

e) $E(X) = \mu = n * p = 8 * 0.5 = 4$ & $Var(X) = \sigma^2 = n * p * q = 8 * 0.5 * 0.5 = 2$

probability distribution of the random variable: **Number of heads**

X: is **binomial distribution**

$$P(X = x) = \binom{8}{x} 0.5^{x}\, 0.5^{8-x} \quad , \qquad x = 0, 1, 2, \dots, 8$$

X	P(x)
0	0.003906
1	0.03125
2	0.109375
3	0.21875
4	0.273438
5	0.21875
6	0.109375
7	0.03125
8	0.003906

Example:

Assuming the experiment of rolling a die **7 times,**

a) Obtain the **probability distribution** of the variable: of getting a face number greater than 4 (i.e: Getting numbers 5 or 6).

b) Obtain the probability of getting a face number greater than 4, 6 times.

c) find E(X) & Var(X)

Solve:

a) Assuming the **success** in this experiment is to get a face number (greater than 4) with probability of ($\boldsymbol{p} = \frac{2}{6}$) and we repeated (rolling a die) 7 times >>> $\boldsymbol{n} = \mathbf{7}$

X: is **binomial distribution**.$\boldsymbol{p} = \frac{2}{6}$, $\boldsymbol{q} = \frac{4}{6}$ $and\ \boldsymbol{n} = 7$: $\boldsymbol{P(X = x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, 2, \ldots, n}$

$$\boldsymbol{P(X = x) = \binom{7}{x}\left(\frac{2}{6}\right)^x \left(\frac{4}{6}\right)^{7-x}, \qquad x = 0, 1, 2, \ldots, 7}$$

b) $\boldsymbol{P(X = 6) = \binom{7}{6}\left(\frac{2}{6}\right)^6 \left(\frac{4}{6}\right)^{7-6} = 7 * 0.001371742 * 0.666666667 = 0.006401463}$

c) $\boldsymbol{E(X) = \mu = n * p = 7 * \left(\frac{2}{6}\right) = \left(\frac{14}{6}\right)}$ & $\boldsymbol{Var(X) = \sigma^2 = n * p * q = 7 * \left(\frac{2}{6}\right) * \left(\frac{4}{6}\right) = \left(\frac{56}{6}\right)}$

probability distribution of the random variable: of getting a face number greater than 4

X: is **binomial distribution**

$$P(X=x)=\binom{7}{x}\left(\frac{2}{6}\right)^{x}\left(\frac{4}{6}\right)^{7-x},\qquad x=0,1,2,\dots,7$$

X	P(x)
0	0.058528
1	0.204847
2	0.30727
3	0.256059
4	0.128029
5	0.038409
6	0.006401
7	0.000457

Definitions: **Bernouilli distribution**

Bernouilli trial: If there is only 1 trial with probability of success *p* and probability of failure *1-p*, this is called a Bernouilli distribution. (special case of the binomial with *n*=1)

Probability of success:

$$P(X=1)=\binom{1}{1}p^1(1-p)^{1-1}=p$$

Probability of failure:

$$P(X=0)=\binom{1}{0}p^0(1-p)^{1-0}=1-p$$

Poisson Distribution

- Poisson distribution is for counts—if events happen at a constant rate over time, the Poisson distribution gives the probability of X number of events occurring in time T.

For a Poisson random variable,

the variance and mean are the same!

- Mean $\mu = \lambda$
- Variance $\sigma^2 = \lambda$
- Standard Deviation $\sigma = \sqrt{\lambda}$

where λ = expected number of hits in a given time period

<u>Example</u>

Arrivals at a bus-stop follow

a Poisson distribution with an <u>average</u> of 4.5 <u>every quarter of an hour</u>.

Obtain a barplot of the distribution (assume a maximum of 20 arrivals in a quarter of an hour) and <u>**calculate**</u> :

the probability of fewer than 3 arrivals in a quarter of an hour.

The probability of fewer than 3 arrivals in a quarter of an hour

=

The probabilities of 0 up to 2 arrivals can be calculated directly from the formula:

$$p(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

with λ =4.5

$$p(0) = \frac{e^{-4.5} 4.5^0}{0!}$$

So p(0) = 0.01111

Similarly p(1)=0.04999 and p(2)=0.11248

So the probability of fewer than 3 arrivals is :

p(0) + p(1) + p(2) =

0.01111+ 0.04999 + 0.11248 =0.17358